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Three Aspects of Mathematical Models for Asymmetric Information in Financial Market

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Declaration

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I confirm that Chapter 2 was jointly co-authored with Assoc Prof. Hao Xing.

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Abstract

The thesis consists of three parts. The first part studies the Glosten-Milgrom model [25] where the risky asset value admits an arbitrary discrete distribution. In contrast to existing results on insider model, the insiders optimal strategy in this model, if it exists, is not of feedback type. Therefore, a weak formulation of equilibrium is proposed. In this weak formulation, the inconspicuous trade theorem still holds, but the optimality for the insiders strategy is not enforced. However, the insider can employ some feedback strategies whose associated expected profit are close to the optimal value, when the order size is small. Moreover, this discrepancy converges to zero when the order size diminishes.

The second part extends Peng's monotone convergence result [37] to backward stochastic differential equations (BSDEs in short) driven by marked point processes. We apply this result to give a stochastic representation to the value function of the insiders problem in the previous part.

The last part studies an optimal trading problem in limit order market with asymmetry information. The market consists of a strategic trader and a group of noisy traders. The strategic trader has private prediction on the fundamental value of a risk asset, and aims to maximise her expected profit. Both types of market participants are allowed to place market and limit orders. We aim to find a trading strategy for the strategic trader who uses both limit and market orders. This is formulated as a stochastic control problem that we characterise in terms of a HJB system. We also provide a numerical algorithm to obtain its solution and prove its convergence. Finally, we consider an example to illustrate the optimal trading strategy of the strategic trader.

Contents

Chapter 1

Introduction

1.1 Trading on informational advantage

One main issue in trading is informational differences. Many trades originate not because individuals have investment or liquidation purposes, but because one agent has or believes she has advanced information about what the price will be in the future. The primary reference in this case is a study by Kyle [32]. In Kyle model, there is an agent denoted by *insider or informed trader* who has strong informational advantage, i.e. she knows the true value of the asset. On the other hand, other market participants do not have this information. The insider tries to optimally adjusts her trading strategy based on the price impact that her trades generate. Back [5] formalises and extends Kyle's model in continuous time to allow any continuous distribution as the asset value.

Another influential model, Glosten-Milgrom [25] model, puts market makers at the centre of the problem trading with counterparties who own advanced information. Market makers need to set a positive bid-ask spread to compensate losses incurred by trading with informed traders even when market makers are competitive and risk-neutral. There exists an equilibrium in which the informed trader plays a mixed strategy.

Back and Baruch [6] connect Kyle and Glosten-Milgrom models by showing that a sequence of the Glosten-Milgrom equilibria converges to the Kyle equilibrium when orders are smaller and arrive more and more frequently. The convergence has been studied recently in a mathematical framework in Çetin and Xing [18]. However, in [6] and [18], the asset value, denoted by $\tilde{v}$, is assumed to have Bernoulli distribution which is quite restricted comparing to general results in Back's paper [5].

In Chapter 2, we consider the Glosten-Milgrom model whose risky asset value $\tilde{v}$ has a discrete distribution. This generalizes the setting in [6] and [18] that $\tilde{v}$ has a Bernoulli distribution. Then we are interested to study whether the Glosten-Milgrom equilibrium still exists when the asset value $\tilde{v}$ has general discrete distributions. In contrast to existing results on the insider model, the insider's optimal strategy in this model, if it exists, is not of feedback type. Therefore, a weak formulation of equilibrium is proposed. In this weak formulation, the inconspicuous trade theorem still holds, but the optimality for the insiders strategy is not enforced. However, the insider can employ some feedback strategies whose associated expected profit are close to the optimal value, when the order size is small. Moreover, this discrepancy converges to zero when the order size diminishes.

1.2 Backward stochastic differential equation

Backward stochastic differential equations (BSDEs in short) were first introduced by Bismut [11]. Then Pardoux and Peng [36] considered non-linear BSDEs in a Brownian setting. A solution to a BSDE is a pair of adapted processes (Y, Z) which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T.$$

When the function g is Lipschitz continuous with respect to (y, z) and $\mathbb{E}[|\xi|^2] < \infty$, the existence and uniqueness of the solution have been proved in [36].

Going beyond the Brownian framework, Barles et al. [8] consider a BSDE driven by a Brownian motion and an independent Poisson random measure. Since then, many generalizations have been considered. In particular, Confortola and Fuhrman [20] build up a connection between optimal control problems and a class of BSDEs, both driven by a marked point process. They show that there exists a unique solution to a BSDE which identifies the value function for the optimal control problem.

Peng [37] studies a limit convergence of BSDEs driven by a Brownian motion such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + C_T - C_t, \quad 0 \le t \le T, \tag{1.2.1}$$

where C is a càdlàg increasing process with $C_0 = 0$ and $\mathbb{E}[|C_T|^2] < \infty$.

Definition 1.2.1 Given a non-decreasing process C, a terminal value ξ and a generator g, if the pair (Y, Z) solves (1.2.1), then we call Y a *supersolution* of BSDE with generator g or simply called *g-supersolution* on $[0, T]$. In particular, when $C \equiv 0$, Y is called a *g-solution* on $[0, T]$.

Peng proved a limit theorem that if a sequence of supersolutions Y^n increasingly converging to Y with $\mathbb{E}\Big[\sup_t |Y_t|^2\Big] < \infty$, then Y itself is a càdlàg supersolution of the same BSDE. As an application, he constructed a family of penalised BSDEs, which converges to the smallest supersolution of a BSDE with a constraint.

In Chapter 2, we consider the value function of insider's optimal trading problem. There has a boundary layer in value function as the time goes to the terminal time. The source of the boundary layer can be determined by a convergence in value functions as the trading intensities of the insider goes to infinity. Since the value functions can be represented by a sequence of penalised BSDEs driven by marked point processes, we analyse the convergence in BSDEs. In Chapter 3, we first extend Confortola and Fuhrman [20] to prove the well-posedness of penalised BSDEs driven by marked point processes. Then we extend Peng's [37] monotone convergence theorem from Brownian setting to a marked point process setting. Finally, we analyse the convergence in BSDEs to determine the source of the boundary layer.

1.3 Trading on limit order book

In Chapter 2, we focus on a control problem which relays on market orders only. However, in practise, agents frequently post limit orders as they are cheaper than market orders. Hence, in Chapter 4, we study a problem how a strategy trader uses her private prediction on the asset and trades both market and limit orders to maximise her trading profits.

Since there are very few papers considering asymmetric information in limit order book, we borrow ideas from the literature of optimal execution in limit order book. For instance, Back and Pedersen [7] show that a strategic trader uses her private prediction gradually and completely in whole trading period. Avellaneda and Stoikov [4] investigate that a market marker maximises terminal wealth by trading in and out of positions using limit orders. Guilbaud and Pham [27] study that a market maker is to maximize her expected utility from revenue over a short term horizon by a trade-off between limit and market orders, while controlling her inventory position. In this case, the agent searches for both an optimal trading size and a sequence of optimal stopping times at which to execute market orders.

In our model, there are two types of agents, noisy traders and a strategic trader, all of whom are risk neutral but they have different information. Both market participants are allowed to place market and limit orders. The strategic trader has some private signal, which is her private valuation prediction of the asset price. She uses the private prediction to trade in the market and maximise her expected profit. Rest market participants are aggregated to noise traders.

Moreover, for market orders, we assume that the size of buy or sell order k takes values from the set of integers $K_m = \{1, \ldots, \bar{m}\}$ where the subscript m stands for market orders. These buy or sell orders are modelled by Poisson processes. The strategic trader controls intensities of Poisson processes, i.e. the speed of placing market orders. On the other hand, for limit orders, they are executed when they are filled by incoming counterpart market orders. After previous execution of limit orders, the strategic trader cancels unexecuted orders and submits new limit orders to wait for next arrival of market orders. We assume that after each arrival of market orders, the strategic trader can submit limit orders, either on buy or sell side, i.e. limit orders are submitted right after the last execution. If there is non-execution or partial execution of limit orders, she will cancel the whole or rest orders immediately before placing new limit orders to wait a next execution. Furthermore, for simplicity of the model, we assume that she always submits limit orders at best bid or ask and those have highest priority to be executed compared to other outstanding limit orders. Hence, according to our assumption, the strategic trader needs to control the size of limit orders either on buy or sell side at each submission. In addition, we also consider the price impact of limit and market orders in our model.

We formulate the problem as a stochastic control problem and prove that the value function of the strategic trader is a solution to this HJB equation. We also investigate numerically the strategic trader's optimal strategy in a market where limit and market orders have two sizes, small and large. We consider five different scenarios depending on sizes of orders allowed to trade by strategic and noise traders. Our numerical solution shows that the strategic trader will place limit and market buy orders when the magnitude of mispricing, which is the difference between her private prediction on the asset and the current trading price, is higher than a threshold. In certain cases, she may even employ a "round trip" strategy to first submit limit sell orders to push price down, and subsequently uses market buy orders to make profit on low market price. In this round trip of trade, the profits from the market buy are still more than losses from the limit sell.

Chapter 2

Asymptotic Glosten-Milgrom equilibrium

2.1 Introduction

In the theory of market microstructure, two models, due to Kyle [32] and Glosten-Milgrom [25], are particularly influential. In the Kyle model, buy and sell orders are batched together by a market maker, who sets a unique price at each auction date. In the Glosten-Milgrom model, buy and sell orders are executed by the market maker individually, hence bid and ask prices appear naturally. In both models[1], an informed agent (insider) trades to maximize her expected profit utilizing her private information on the asset fundamental value, while another group of noise traders trade independently of the fundamental value. The cumulative demand of these noise traders is modelled by a Brownian motion in Kyle model, cf. [5], and by the difference of two independent Poisson processes, whose jump size is scaled by the order size, in the Glosten-Milgrom model.

When the fundamental value, described by a random variable $\tilde{v}$, has an arbitrary continuous distribution[2], Back [5] establishes a unique equilibrium between the insider and the market maker. Moreover, the cumulative demand process in the equilibrium connects elegantly to the theory of filtration enlargement, cf. [35]. However much less is known about equilibrium in the Glosten-Milgrom model. Back and Baruch [6] consider a Bernoulli distributed $\tilde{v}$. In this case, the insider's optimal strategy is constructed in [18]. Equilibrium with general distribution of $\tilde{v}$, as Cho [19] puts it, "will be a great challenge to consider".

In this paper, we consider the Glosten-Milgrom model whose risky asset value $\tilde{v}$ has a discrete

[1] A profit maximizing informed agent is introduced in the Glosten-Milgrom model in [6]

[2] Models with discrete distributed $\tilde{v}$ can be studied similarly as in [5].

distribution:

$$\mathbb{P}(\tilde{v} = v_n) = p_n, \quad n = 1, \cdots, N, \tag{2.1.1}$$

where $N \in \mathbb{N} \cup \{\infty\}$, $(v_n)_{n=1,\cdots,N}$ is an increasing sequence and $p_n \in (0,1)$ with $\sum_{n=1}^{N} p_n = 1$. This generalizes the setting in [6] where $N = 2$ is considered, i.e., $\tilde{v}$ has a Bernoulli distribution.

In models of insider trading, *inconspicuous trade theorem* is commonly observed, cf. e.g., [32], [5], [7], [6], [14], and [15] for equilibria of Kyle type, and [18] for the Glosten-Milgrom equilibrium with Bernoulli distributed fundamental value. The inconspicuous trade theorem states, when the insider is trading optimally in equilibrium, the cumulative net orders from both insider and noise traders have the same distribution as the net orders from noise traders, i.e., the insider is able to hide her trades among noise trades. As a consequence, this allows the market maker to set the trading price only considering current cumulative noise trades. Moreover, in all aforementioned studies, the insider's optimal strategy is of *feedback form*, which only depends on the current cumulative total order. This functional form is associated to optimizers of the Hamilton-Jacobi-Bellman (HJB) equation for the insider's optimization problem. However the situation is dramatically different in the Glosten-Milgrom model with N in (2.1.1) at least 3. Theorem 2.2.6 below shows that, given aforementioned pricing mechanism, the insider's optimal strategy, if exists, does *not* correspond to optimizers of the HJB equation. The result is consequence of the difference between bid and ask prices in the Glosten-Milgrom model, which is contrast to the unique price in the Kyle model.

Therefore to establish equilibrium in these Glosten-Milgrom models, we propose a weak formulation of equilibrium in Definition 2.2.11, which is motivated by the convergence of Glosten-Milgrom equilibria to the Kyle equilibrium, as the order size diminishing and the trading intensities increasing to infinity, cf. [6] and [18]. In this weak formulation, the insider still trades to enforce the inconspicuous trading theorem, but the insider's strategy may not be optimal. However, the insider can employ some feedback strategy so that the loss to her expected profit (compared to the optimal value) is small for small order size. Moreover this gap converges to zero when the order size vanishes. We call this weak formulation *asymptotic Glosten-Milgrom equilibrium* and establish its existence in Theorem 2.2.12.

In the asymptotic Glosten-Milgrom equilibrium, the insider's strategy is constructed explicitly in section 2.5, using a similar construction as in [18]. Using this strategy, the insider trades towards a middle level of an interval, driving the total demand process into this interval at the terminal date. This bridge behaviour is widely observed in the aforementioned studies on insider trading. On the other hand, the insider's strategy is of feedback form. Hence the insider can determine her trading intensity only using the current cumulative total demand. Moreover, as order size diminishes, the family of suboptimal strategies converge to the optimal strategy in Kyle model, cf. Theorem 2.2.13. In such an asymptotic Glosten-Milgrom equilibrium, the insider loses some expected profit.

The expression of this profit loss is quite interesting mathematically: it is the difference of two stochastic integrals with respect to (scaled) Poisson occupation time. As the order size vanishes, both integrals converge to the same stochastic integral with respect to Brownian local time, hence their difference vanishes.

This chapter is organized as follows. Main results are presented in section 2.2. The mismatch between insider's optimal strategy and optimizers for the HJB equation is proved in section 2.3. Then a family of suboptimal strategies are characterized and constructed in Sections 2.4 and 2.5. Finally the existence of asymptotic equilibrium is established in section 2.6 and a technical result is proved in Appendix.

2.2 Main results

2.2.1 The model

We consider a continuous time market for a risky and a risk free asset. The risk free interest rate is normalized to 0, i.e., the risk free asset is regarded as the numéraire. We assume that the *fundamental value* of the risky asset $\tilde{v}$ has a discrete distribution of type (2.1.1). This fundamental value will be revealed to all market participants at a finite time horizon, say 1, at which point the market will terminate.

The micro-structure of the market and the interaction of market participants are modelled similarly to [6] which we recall below. There are three types of agents: uninformed/noise traders, an informed trader/insider, and a market maker, all of whom are risk neutral. These agents share the same view toward future randomness of the market, but they possess different information. Therefore, the probability space $(\Omega, \mathbb{P})$ with different filtration accommodates the following processes:

- *Noise traders* trade for liquidity or hedging reasons which are independent of the fundamental value $\tilde{v}$. The cumulative demand Z is described by the difference of two independent jump processes Z^B and Z^S which are the cumulative buy and sell orders, respectively. Therefore $Z = Z^B - Z^S$ and it is independent of $\tilde{v}$. Noise traders only submit orders of fixed sized δ every time they trade. As in [6], Z^B/δ and Z^S/δ are assumed to be independent Poisson processes with constant intensity β. Let $(\mathscr{F}^Z_t)_{t\in[0,1]}$ be the smallest filtration generated by Z and satisfying the usual conditions. Then $(\mathscr{F}^Z_t)_{t\in[0,1]}$ describes the information structure of noise traders.

- The *insider* knows the fundamental value $\tilde{v}$ at time 0 and observes the market price for the risky asset between time 0 and 1. The insider also submits orders of fixed size δ in every trade and tries to maximize her expected profit. The cumulative demand from the insider is denoted by $X := X^B - X^S$ where X^B and X^S are cumulative buy and sell orders respectively. Since the insider observes the market price of the risky asset, she can back out the dynamics of noise orders, cf. discussions after Definition 2.2.1. Therefore the information structure of the insider $\mathscr{F}_t^I$ includes $\mathscr{F}_t^Z$ and $\sigma(\tilde{v})$, for any $t \in [0,1]$.

- A competitive *market maker* only observes the aggregation of the informed and noise trades, so he cannot distinguish between informed and noise trades. Given $Y := X + Z$, the information of the market maker is $(\mathscr{F}_t^Y)_{t\in[0,1]}$ generated by Y and satisfying the usual conditions. As the market maker is risk neutral, the competition will force him to set the market price as $\mathbb{E}[\tilde{v}|\mathscr{F}_t^Y]$, $t \in [0,1]$.

In order to define equilibrium in the market, let us first describe admissible actions for the market maker and the insider. The market maker looks for a Markovian pricing mechanism, in which the price of the risky asset at time t is set using cumulative order Y_t and a pricing rule p.

Definition 2.2.1 A function $p : \delta\mathbb{Z} \times [0,1] \to \mathbb{R}$ is a *pricing rule* if

i) $y \mapsto p(y,t)$ is strictly increasing for each $t \in [0,1)$;

ii) $\lim_{y\to-\infty} p(y,t) = v_1$ and $\lim_{y\to\infty} p(y,t) = v_N$ for each $t \in [0,1]$;

iii) $t \mapsto p(y,t)$ is continuous for each $y \in \delta\mathbb{Z}$.

The monotonicity of $y \mapsto p(y,t)$ in i) is natural. It implies that the market price is higher whenever the demand is higher. Moreover, because of the monotonicity, the insider fully observes the uninformed orders Z by inverting the price process and subtracting her orders from the total orders. Item ii) means that the range of the pricing rule is wide enough to price in every possibility of fundamental value. The insider trades to maximize her expected profit. Her admissible strategy is defined as follows:

Definition 2.2.2 The strategy $(X^B, X^S; \mathscr{F}^I)$ is *admissible*, if

i) $\mathscr{F}^I$ is a filtration satisfying the usual conditions and generated by $\sigma(\tilde{v})$, $\mathscr{F}^Z$, and $\mathcal{H}$, where $(\mathcal{H}_t)_{t\in[0,1]}$ is a filtration independent of $\tilde{v}$ and $\mathscr{F}^Z$;

ii) X^B and X^S, with $X_0^B = X_0^S = 0$, are $\mathscr{F}^I$-adapted and integrable[3] increasing point processes with jump size δ;

iii) the $(\mathscr{F}^I, \mathbb{P})$-dual predictable projections of X^B and X^S are absolutely continuous with respect to time, hence X^B and X^S admit $\mathscr{F}^I-$intensities θ^B and θ^S, respectively;

[3]That is, $\mathbb{E}[X_1^B]$ and $\mathbb{E}[X_1^S]$ are both finite.

iv) $\mathbb{E}\left[\int_0^1 |p(Y_t,t)|\,|dX_t^i - \delta\theta_t^i dt|\right] < \infty$, for $i \in \{B, S\}$ and the pricing rule p fixed by the market maker. Here $|X^i - \int_0^\cdot \delta\theta^i dt|$ is the variation of the compensated point process.

This set of admissible strategies is similar to [18, Definition 2.2]. Item i) assumes that the insider is allowed to possess additional information $\mathcal{H}$, independent of $\tilde{v}$ and $\mathscr{F}^Z$, which she uses to generate her mixed strategy. Item iv) implies $\delta\mathbb{E}[\int_0^1 |p(Y_t,t)|\,\theta_t^i dt] < \infty$, hence the expected profit of the insider is finite. Item ii) does not exclude the insider trading at the same time with noise traders. When the insider submits an order at the same time when an uniformed order arrives but in the opposite direction, assuming the market maker only observes the net demand implies that such pair of trades goes unnoticed by the market maker. This pair of opposite orders will be executed without a need for a market maker. Hence the market maker only knows the transaction when there is a need for him. Henceforth, when the insider makes a trade at the same time with an uninformed trader but in an opposite direction, we say the insider *cancels* the noise trades. On the other hand, item ii) also allows the insider to trade at the same time with noise traders in the same direction. We call that the insider *tops up* noise orders in this situation. However, the insider does not submit such orders in equilibrium, even when equilibrium is defined in a weak sense, cf. Remark 2.4.6 below. The assumption that the insider is allowed to trade at the same time as noise traders is different from assumptions for Kyle model where insider's strategy is predictable. This additional freedom for insider is not the source for Theorem 2.2.6 below, which states optimizers for the insider's HJB equation do not correspond to the optimal strategy; see Remark 2.2.8 below.

As described in the last paragraph, the insider's cumulative buy orders may consist of three components: $X^{B,B}$ arrives at different time than those of Z^B, $X^{B,T}$ arrives at the same time as some orders of Z^B, and $X^{B,S}$ cancels some orders of Z^S. Sell orders X^S are defined analogously. Therefore $X^B = X^{B,B} + X^{B,T} + X^{B,S}$ and $X^S = X^{S,S} + X^{S,T} + X^{S,B}$.

As mentioned earlier, the insider aims to maximize her expected profit. Given an admissible trading strategy $X = X^B - X^S$ the associated profit at time 1 of the insider is given by

$$\int_0^1 X_{t-} dp(Y_t,t) + (\tilde{v} - p(Y_1,1))X_1.$$

The last term appears due to a potential discrepancy between the market price and the liquidation value. Since X is of finite variation and $X_0 = 0$, applying integration by parts rewrites the profit as

$$\begin{aligned}
&\int_0^1 (\tilde{v} - p(Y_t,t))\,dX_t^B - \int_0^1 (\tilde{v} - p(Y_t,t))\,dX_t^S \\
&= \int_0^1 (\tilde{v} - p(Y_{t-}+\delta,t))\,dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-}+2\delta,t))\,dX^{B,T} \\
&\quad + \int_0^1 (\tilde{v} - p(Y_{t-},t))\,dX_t^{B,S} - \int_0^1 (\tilde{v} - p(Y_{t-}-\delta,t))\,dX_t^{S,S} \\
&\quad - \int_0^1 (\tilde{v} - p(Y_{t-}-2\delta,t))\,dX^{S,T} - \int_0^1 (\tilde{v} - p(Y_{t-},t))\,dX_t^{S,B},
\end{aligned}$$

where Y increases (resp. decreases) δ when $X^{B,B}$ (resp. $X^{S,S}$) jumps by δ, Y increases (resp. decreases) 2δ when $X^{B,T}$ (resp. $X^{S,T}$) jumps at the same time with Z^B (resp. Z^S), and Y is unchanged when $X^{S,B}$ (resp. $X^{B,S}$) jumps at the same time with Z^B (resp. Z^S). Define

$$a(y,t) := p(y+\delta,t) \qquad \text{and} \qquad b(y,t) := p(y-\delta,t),$$

which can be viewed as ask and bid prices respectively. Then the expected profit of the insider conditional on her information can be expressed as

$$\begin{aligned}\mathbb{E}\Bigg[&\int_0^1 (\tilde{v} - a(Y_{t-},t))\, dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-},t))\, dX_t^{B,S} \\ &+ \int_0^1 (\tilde{v} - a(Y_{t-}+\delta,t))\, dX_t^{B,T} - \int_0^1 (\tilde{v} - b(Y_{t-}-\delta,t))\, dX_t^{S,T} \\ &- \int_0^1 (\tilde{v} - b(Y_{t-},t))\, dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-},t))\, dX_t^{S,B} \Big| \tilde{v}\Bigg].\end{aligned} \tag{2.2.1}$$

Having described the market structure, an equilibrium between the market maker and the insider is defined as in [6]:

Definition 2.2.3 A *Glosten-Milgrom equilibrium* is a quadruplet $(p, X^B, X^S, \mathscr{F}^I)$ such that

i) given $(X^B, X^S; \mathscr{F}^I)$, p is a rational pricing rule, i.e., $p(Y_t,t) = \mathbb{E}[\tilde{v}|\mathscr{F}_t^Y]$ for $t \in [0,1]$;

ii) given p, $(X^B, X^S; \mathscr{F}^I)$ is an admissible strategy maximizing (2.2.1).

When $N = 2$, [18] establishes the existence of Glosten-Milgrom equilibria. In equilibrium the pricing rule is

$$p(y,t) = \mathbb{E}^{\mathbb{P}^y}\left[P(Z_{1-t})\right], \quad (y,t) \in \delta\mathbb{Z} \times [0,1]. \tag{2.2.2}$$

Here $\mathbb{P}^y$ is a probability measure under which Z is the difference of two independent Poisson processes and $\mathbb{P}^y(Z_0 = y) = 1$. P is a nondecreasing function such that $P(Z_1)$ has the same distribution as $\tilde{v}$. Moreover the optimal strategy of the insider are given by jump processes $X^{i,j}$, $i \in \{B,S\}$ and $j \in \{B,T,S\}$, with intensities $\delta\, \theta^{i,j}(Y_{t-},t)$, $t \in [0,1]$. These intensities are deterministic functions of the state variable Y, hence this control strategy is a *feedback control* and it corresponds to optimizers of insider's HJB equation. However, when $N \geq 3$, Theorem 2.2.6 below shows that, given the pricing rule (2.2.2), the optimal strategy *does not* correspond to optimizers in the HJB equation, for some values of $\tilde{v}$. This result is surprising, because it is contrast to existing results in the Kyle and Glosten-Milgrom equilibrium; cf. [32], [5], [7], [6], [14], [15], and [18]. This mismatch roots in the discrete state space of the demand process in the Glosten-Milgrom model. The discrete state space yields different bid and ask prices, which is contrast to the unique price in the Kyle model. See Remark 2.2.7 below for more discussion.

2.2.2 Nonexistence of a feedback optimal control

To state aforementioned result, we introduce several additional notations. For each $\delta > 0$, let $\Omega^\delta = \mathbb{D}([0,1], \delta\mathbb{Z})$ be the space of $\delta\mathbb{Z}$-valued càdlàg functions on $[0,1]$ with coordinate process Z^δ, $(\mathscr{F}_t^{Z,\delta})_{t\in[0,1]}$ is the minimal right continuous and complete filtration generated by Z^δ, and $\mathbb{P}^\delta$ is the probability measure under which Z^δ is the difference of two independent Poisson processes starting from 0 with the same jump size δ and intensity β^δ. We denote by $\mathbb{P}^{\delta,y}$ the probability measure under which $Z_0^\delta = y$ a.s.. Henceforth, the superscript δ indicates the trading size in the Glosten-Milgrom model.

For the fundamental value $\tilde{v}^\delta$, let us first consider the following family of distributions.

Assumption 2.2.4 Given $\tilde{v}^\delta$ of type (2.1.1), there exists a $\delta\mathbb{Z}\cup\{-\infty,\infty\}-$valued strictly increasing sequence $(a_n^\delta)_{n=1,\cdots,N+1}$[4] with $a_1^\delta = -\infty$, $a_{N+1}^\delta = \infty$, and $\bigcup_{n=1}^N [a_n^\delta,\, a_{n+1}^\delta) = \delta\mathbb{Z}\cup\{-\infty\}$, such that

$$\mathbb{P}(\tilde{v}^\delta = v_n) = \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right), \qquad n = 1, \cdots, N. \tag{2.2.3}$$

For any $\tilde{v}$ with discrete distribution (2.1.1), Lemma 2.6.1 below shows there exists a sequence $(\tilde{v}^\delta)_{\delta>0}$, each satisfies Assumption 2.2.4 and converges to $\tilde{v}$ in law as $\delta \downarrow 0$. Therefore any $\tilde{v}$ of type (2.1.1) can be approximated by a $\tilde{v}^\delta$ satisfying Assumption 2.2.4. Given $\tilde{v}^\delta$ satisfying Assumption 2.2.4, define

$$h_n^\delta(y,t) := \mathbb{P}^{\delta,y}\left(Z_{1-t}^\delta \in [a_n^\delta, a_{n+1}^\delta)\right), \quad y \in \delta\mathbb{Z}, \quad t \in [0,1], \quad n \in \{1, \cdots, N\}, \tag{2.2.4}$$

and

$$p^\delta(y,t) := \sum_{n=1}^N v_n h_n^\delta(y,t) = \mathbb{E}^{\delta,y}\left[P(Z_{1-t}^\delta)\right], \tag{2.2.5}$$

where the expectation is taken under $\mathbb{P}^{\delta,y}$ and

$$P(y) = v_n, \qquad \text{when } y \in [a_n^\delta, a_{n+1}^\delta). \tag{2.2.6}$$

Then (2.2.3) implies that $\tilde{v}^\delta$ and $P(Z_1^\delta)$ have the same distribution. If p^δ is chosen as the pricing rule, it has the same form as in (2.2.2). Finally we impose a technical condition on p^δ. This assumption is clearly satisfied when N is finite.

Assumption 2.2.5 There exist positive constants C and n such that $|p^\delta(y,t)| \le C(1+|y|^n)$ for any $(y,t) \in \delta\mathbb{Z}\times[0,1]$.

Given the pricing rule (2.2.5), let us first study the insider's optimization problem and derive the associated HJB equation via a heuristic argument. In this derivation, the superscript δ is omitted

[4]When $N = \infty$, $N+1 = \infty$.

to simplify notation. Definition 2.2.2 iii) implies that $X^{i,j} - \delta \int_0^\cdot \theta_r^{i,j} dr$ defines an $\mathscr{F}^I$-martingale for $i \in \{B, S\}$ and $j \in \{B, T, S\}$. On the other hand, Definition 2.2.2 iv) and [13, Chapter I, T6] combined imply that $\int_0^\cdot (\tilde{v} - p(Y_{r-} + \delta, r))(dX_r^{B,B} - \delta\theta_r^{B,B} dr) = \int_0^\cdot (\tilde{v} - p(Y_r, r))(dX_r^{B,B} - \delta\theta_r^{B,B} dr)$ is an $\mathscr{F}^I$-martingale. Similar argument applied to other terms allows us to rewrite (2.2.1) as

$$\begin{aligned}\delta\mathbb{E}\Big[&\int_0^1 (\tilde{v} - p(Y_{r-} + \delta, r))\theta_r^{B,B}\, dr + \int_0^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{B,S}\, dr \\ &+ \int_0^1 (\tilde{v} - p(Y_{r-} + 2\delta, r))\theta_r^{B,T}\, dr - \int_0^1 (\tilde{v} - p(Y_{r-} - \delta, r))\theta_r^{S,S}\, dr \\ &- \int_0^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{S,B}\, dr - \int_0^1 (\tilde{v} - p(Y_{r-} - 2\delta, r))\theta_r^{S,T}\, dr \Big| \tilde{v}\Big].\end{aligned}$$

This motivates us to define the following value function for the insider:

$$\begin{aligned}V^\delta(\tilde{v}, y, t) := &\sup_{\theta^{i,j};\, i\in\{B,S\}, j\in\{B,T,S\}} \\ &\delta\,\mathbb{E}\Big[\int_t^1 (\tilde{v} - p(Y_{r-} + \delta, r))\theta_r^{B,B}\, dr + \int_t^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{B,S}\, dr \\ &+ \int_t^1 (\tilde{v} - p(Y_{r-} + 2\delta, r))\theta_r^{B,T}\, dr - \int_t^1 (\tilde{v} - p(Y_{r-} - \delta, r))\theta_r^{S,S}\, dr \\ &- \int_t^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{S,B}\, dr - \int_t^1 (\tilde{v} - p(Y_{r-} - 2\delta, r))\theta_r^{S,T}\, dr \Big| Y_t = y, \tilde{v}\Big],\end{aligned} \tag{2.2.7}$$

for $\tilde{v} = \{v_1, \cdots, v_N\}$, $y \in \delta\mathbb{Z}$, $t \in [0, 1)$. The terminal value of V^δ is defined as $V^\delta(\tilde{v}, y, 1) = \lim_{t\to 1} V^\delta(\tilde{v}, y, t)$ [5]. Lemma 2.3.2 and Proposition 2.4.4 below show that the optimization problem in (2.2.7) is well defined and nontrivial, i.e., $0 < V^\delta < \infty$, for each $\delta > 0$. Let us now derive the HJB equation which V^δ satisfies via a heuristic argument. Since positive (resp. negative) part of Y is $Y^B := X^{B,B} + X^{B,T} + Z^B - X^{S,B}$ (resp. $Y^S := X^{S,S} + X^{S,T} + Z^S - X^{B,S}$). Hence $Y^B - \delta \int_0^\cdot (\beta - \theta_r^{S,B} - \theta_r^{B,T})\, dr - \delta \int_0^\cdot \theta_r^{B,B}\, dr - 2\delta \int_0^\cdot \theta_r^{B,T}\, dr$ (resp. $Y^S - \delta \int_0^\cdot (\beta - \theta_r^{B,S} - \theta_r^{S,T})\, dr - \delta \int_0^\cdot \theta_r^{S,S}\, dr - 2\delta \int_0^t \theta_r^{S,T}\, dr$) is an $\mathscr{F}^I$-martingale.[6] Then applying Itô's formula to $V^\delta(\tilde{v}, Y_r, r)$ and employing the standard dynamic programming arguments yield the following formal HJB equation for V^δ:

$$-V_t(v_n, y, t) - H(v_n, y, t, V) = 0, \quad n \in \{1, \cdots, N\}, \quad (y, t) \in \delta\mathbb{Z} \times [0, 1), \tag{2.2.8}$$

where the Hamilton H is defined as (the $\tilde{v}$ argument is omitted in H to simplify notation)

$$H(v_n, y, t, V) :=(V(y + \delta, t) - 2V(y, t) + V(y - \delta, t))\beta$$

---

[5] Since the set of admissible control is unbounded, the HJB equation associated to (2.2.7) usually admits a *boundary layer*, i.e., $\lim_{t\to 1} V^\delta(\tilde{v}, y, t)$ is not identically zero even if there is no terminal profit in (2.2.1). Such phenomenon also shows up in Kyle model, see [5].

[6] As discussed after Definition 2.2.2, the set of jumps of $X^{B,S}$ and $X^{S,T}$ (resp. $X^{S,B}$ and $X^{B,T}$) arrive at the same time as some jumps of Z^S (resp. Z^B), then we necessarily have $\theta^{B,S} + \theta^{S,T} \le \beta$ (resp. $\theta^{S,B} + \theta^{B,T} \le \beta$).

$$
\begin{aligned}
&+ \sup_{\theta^{B,B}\geq 0} \big[V(y+\delta,t) - V(y,t) + (v_n - p(y+\delta,t))\delta\big]\theta^{B,B} \\
&+ \sup_{\theta^{B,T}\geq 0} \big[V(y+2\delta,t) - V(y+\delta,t) + (v_n - p(y+2\delta,t))\delta\big]\theta^{B,T} \\
&+ \sup_{\theta^{B,S}\geq 0} \big[V(y,t) - V(y-\delta,t) + (v_n - p(y,t))\delta\big]\theta^{B,S} \qquad (2.2.9)\\
&+ \sup_{\theta^{S,S}\geq 0} \big[V(y-\delta,t) - V(y,t) - (v_n - p(y-\delta,t))\delta\big]\theta^{S,S} \\
&+ \sup_{\theta^{S,T}\geq 0} \big[V(y-2\delta,t) - V(y-\delta,t) - (v_n - p(y-2\delta,t))\delta\big]\theta^{S,T} \\
&+ \sup_{\theta^{S,B}\geq 0} \big[V(y,t) - V(y+\delta,t) - (v_n - p(y,t))\delta\big]\theta^{S,B}.
\end{aligned}
$$

Optimizers $\theta^{i,j}$, $i \in \{B,S\}$ and $j \in \{B,T,S\}$, in (2.2.9), are deterministic functions of v_n, y and t, hence they are of feedback form. They are expected to be the optimal control intensities for (2.2.7). This is indeed the case in many existing results in Kyle model and Glosten-Milgrom model (with $N = 2$), compare [32], [5], [7], [6], and [18]. However, when $N \geq 3$ in the Glosten-Milgrom model, the following theorem shows any optimizers in (2.2.9) are *not* the optimal intensities when $\tilde{v}$ is neither v_1 nor v_N.

Theorem 2.2.6 *Let* $N \geq 3$ *and* $\tilde{v}^\delta$ *satisfy Assumption 2.2.4. Let* p^δ *in* (2.2.5) *be the pricing rule and satisfy Assumption 2.2.5. Then any optimizers* $\theta^{i,j}(y,t)$*,* $i \in \{B,S\}, j \in \{B,T,S\}$ *and* $(y,t) \in \delta\mathbb{Z} \times [0,1)$*, for* (2.2.9) *are not the optimal strategy for* (2.2.7) *when* $\tilde{v}^\delta = v_n$ *for* $1 < n < N$*.*

Remark 2.2.7 When $\tilde{v}^\delta = v_1$ (resp. v_N), the insider knows the risky asset is always over-priced (resp. under-priced). Hence she always sells (resp. buys) in equilibrium. This situation is exactly the same as [18]. When $\tilde{v}^\delta$ is neither minimal nor maximal, let us briefly describe the proof of Theorem 2.2.6 here. To ensure (2.2.8) to be wellposed, H must be finite for all $(y,t) \in \delta\mathbb{Z} \times [0,1)$. Hence

$$
(p(y,t) - v_n)\delta \leq V(y+\delta,t) - V(y,t) \leq (p(y+\delta,t) - v_n)\delta, \qquad \text{for all } (y,t) \in \delta\mathbb{Z} \times [0,1), \quad (2.2.10)
$$

where the second inequality comes from the first three maximization in (2.2.9) and the first inequality comes from the last three. Since $V > 0$, $\theta^{i,j} \equiv 0$, $i \in \{B,S\}$ and $j \in \{B,T,S\}$, in (2.2.9) does not correspond to the optimal strategy. Hence there must exist (y_0, t_0) such that one inequality in (2.2.10), say the first one, is an equality. However, in this case, the discrete state space forces the first inequality to be an equality for *all* $(y,t) \in \delta\mathbb{Z} \times [0,1)$, which implies the second inequality in (2.2.10) is strict for all (y,t), due to $p(y+\delta,t) > p(y,t)$. Therefore the optimizers in the first three maximization in (2.2.9) must be identically zero, which means the associated point process X does not have positive jumps. On the other hand, the dynamic programming principle and the boundary layer of (2.2.8) at $t = 1$ force $Y_1 = Z_1 + X_1 \in [a_n^\delta + \delta, a_{n+1}^\delta]$ a.s.. This can never happen when X does not have positive jumps. Therefore, Theorem 2.2.6 is the joint effort of the discrete state space and the boundary layer of the HJB equation.

Remark 2.2.8 The statement of Theorem 2.2.6 remains valid when the insider is prohibited from trading with noise traders at the same time; i.e., $X^{B,T}, X^{B,S}, X^{S,T}, X^{S,B}$ are all zero. In this case, the second, third, fifth and sixth maximization do not present in (2.2.9). However, the first and fourth maximization therein still lead to (2.2.10). Hence the same argument as in the previous remark still applies.

Remark 2.2.9 Examples of control problems without optimal feedback control exist in literature of the optimal control theory, cf., e.g. [43, Chapter 3, pp. 246] and [34, Example 1.1]. In these cases, notion of *relaxed control* is employed to prove the existence of a relaxed optimal control, cf. [34] and references therein. For the insider's optimization problem, instead of $\{\theta : \delta\mathbb{Z} \times [0,1] \to \mathbb{R}_+\}$, the control set can be relaxed to $\{\theta : \delta\mathbb{Z} \times [0,1] \to \mathcal{M}^1(\mathbb{R}_+)\}$, where $\mathcal{M}^1(\mathbb{R}_+)$ is the set of all probability measures in $\mathbb{R}_+$. It is interesting to investigate whether (2.2.7) admits an optimal control in this relaxed set. We leave this topic to future studies.

2.2.3 Asymptotic Glosten-Milgrom equilibrium

To establish equilibrium of Glosten-Milgrom type when the risky asset $\tilde{v}$ has general discrete distribution (2.1.1) with $N \geq 3$, we introduce a weak form of equilibrium in what follows. To motivate this definition, we recall the convergence of Glosten-Milgrom equilibria as the order size decreasing to zero and intensity of noise trades increasing to infinity, cf. [6, Theorem 3] and [18, Theorem 5.3]:

Proposition 2.2.10 *For any Bernoulli distributed $\tilde{v}$ (i.e. $N = 2$ in (2.1.1)), there exists a sequence of Bernoulli distributed random variables $\tilde{v}^\delta$ such that*

i) $\tilde{v}^\delta$ converges to $\tilde{v}$ in law as $\delta \downarrow 0$;

ii) For each $\delta > 0$, model with $\tilde{v}^\delta$ as the fundamental value of the risky asset admits a Glosten-Milgrom equilibrium $(p^\delta, X^{B,\delta}, X^{S,\delta}, \mathscr{F}^{I,\delta})$;

iii) When the intensity of Poisson process is given by $\beta^\delta := (2\delta^2)^{-1}$, $X^{B,\delta} - X^{S,\delta} \xrightarrow{\mathcal{L}} X^0$, as $\delta \downarrow 0$, where X^0 is the optimal strategy in the Kyle model and $\xrightarrow{\mathcal{L}}$ represents the weak convergence of stochastic processes[7].

This result motivates us to define the following weak form of Glosten-Milgrom equilibrium:

[7] Refer to [10] or [30] for the definition of weak convergence of stochastic processes.

Definition 2.2.11 For any $\tilde{v}$ with discrete distribution (2.1.1), an *asymptotic Glosten-Milgrom equilibrium* is a sequence $(\tilde{v}^\delta, p^\delta, X^{B,\delta}, X^{S,\delta}, \mathscr{F}^{I,\delta})_{\delta>0}$ such that

i) $\tilde{v}^\delta$ converges to $\tilde{v}$ in law as $\delta \downarrow 0$;

ii) For each $\delta > 0$, given $(\tilde{v}^\delta, X^{B,\delta}, X^{S,\delta}, \mathscr{F}^{I,\delta})$ and set $Y^\delta := Z^\delta + X^{B,\delta} - X^{S,\delta}$, p^δ is a rational pricing rule, i.e., $p^\delta(Y^\delta_t, t) = \mathbb{E}[\tilde{v}^\delta \,|\, \mathscr{F}^{Y^\delta}_t]$ for $t \in [0,1]$;

iii) Given $(\tilde{v}^\delta, p^\delta)$ and $\beta^\delta = (2\delta^2)^{-1}$, let $\mathcal{J}^\delta(X^B, X^S)$ be insider's expected profit associated to the admissible strategy (X^B, X^S). Then

$$\sup_{(X^B,X^S) \text{ admissible}} \mathcal{J}^\delta(X^B, X^S) - \mathcal{J}^\delta(X^{B,\delta}, X^{S,\delta}) \to 0, \quad \text{as } \delta \downarrow 0.$$

In the above definition, rationality of the pricing mechanism is not compromised. However optimality of the insider's strategy is not enforced. Instead, item iii) requires that, when the order size is small, the loss of insider's expected profit by employing the strategy $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{\delta,I})$ is small, comparing to the optimal value. Moreover this discrepancy converges to zero when the order size vanishes. Therefore if the insider is willing to give up a small amount of expected profit, she can employ strategy $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})$ to establish a suboptimal equilibrium. The following result establishes the existence of equilibrium in the above weak sense:

Theorem 2.2.12 *Assume that $\tilde{v}$ satisfies* (2.1.1) *with $N < \infty$. Then asymptotic Glosten-Milgrom equilibrium exists.*

In this asymptotic equilibrium, the pricing rule is given by (2.2.5). When the order size is δ, the insider employs the strategy $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})$ with $\mathscr{F}^{I,\delta}$-intensities

$$\begin{aligned} &\delta\beta^\delta \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h^\delta_n(Y^\delta_{t-}+\delta, t)}{h^\delta_n(Y^\delta_{t-}, t)} - 1\right]_+ + \delta\beta^\delta \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h^\delta_n(Y^\delta_{t-}-\delta, t)}{h^\delta_n(Y^\delta_{t-}, t)} - 1\right]_-, \\ &\delta\beta^\delta \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h^\delta_n(Y^\delta_{t-}-\delta, t)}{h^\delta_n(Y^\delta_{t-}, t)} - 1\right]_+ + \delta\beta^\delta \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h^\delta_n(Y^\delta_{t-}+\delta, t)}{h^\delta_n(Y^\delta_{t-}, t)} - 1\right]_-, \end{aligned} \tag{2.2.11}$$

respectively. In particular, when the fundamental value is v_n, the insider trades toward the middle level $m^\delta_n := (a^\delta_n + a^\delta_{n+1} - \delta)/2$ of the interval $[a^\delta_n, a^\delta_{n+1})$: when the total demand is less than m^δ_n, the insider only places buy orders by either complementing noise buy orders or canceling some of noise sell orders, when the total demand is larger than m^δ_n, the insider does exactly the opposite. More specifically, Lemma 2.5.2 below shows that $y \mapsto h^\delta_n(y,t)$ is strictly increasing when $y < m^\delta_n$ and strictly decreasing when $y > m^\delta_n$. Therefore, when $Y^\delta_{t-} < m^\delta_n$, (2.2.11) implies that: $X^{B,B,\delta}$ has intensity $\frac{1}{2\delta}\left(\frac{h^\delta_n(Y^\delta_{t-}+\delta,t)}{h^\delta_n(Y^\delta_{t-},t)} - 1\right)$, $X^{B,S,\delta}$ has intensity $\frac{1}{2\delta}\left(1 - \frac{h^\delta_n(Y^\delta_{t-}-\delta,t)}{h^\delta_n(Y^\delta_{t-},t)}\right)$, meanwhile intensities of $X^{S,S,\delta}$ and $X^{S,B,\delta}$ are both zero. When $Y^\delta_{t-} > m^\delta_n$, intensities can be read out from (2.2.11) similarly.

Even though Theorem 2.2.6 remains valid when the insider is prohibited from trading at the same time with noise traders, the strategy constructed above depends on the possibility of canceling orders. However, in this strategy, the insider never tops up noise orders, i.e., $X^{B,T} = X^{S,T} \equiv 0$. This allows the market maker to employ a rational pricing mechanism so that Definition 2.2.11 ii) is satisfied, cf. Remark 2.4.6 below.

The processes $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})$ with intensities (2.2.11) will be constructed explicitly in section 2.5. The insider employs a sequence of independent random variables with uniform distribution on $[0,1]$ to construct her mixed strategy. This sequence of random variables are also independent of Z^δ and $\tilde{v}^\delta$. This construction is a natural extension of [18]. In this construction, whenever a noise order arrives, the insider uses a uniform distributed random variable to decide whether or not submitting an opposite cancelling order. Hence this strategy is adapted to insider's filtration, rather than predictable as in the Kyle model. Such a cancelling strategy is called *input regulation* and has been studied extensively in the queueing theory literature, see eg. [13, Chapter VII, Section 3].

When the fundamental value is v_n and the insider follows the aforementioned strategy, the total demand at time 1 will end up in the interval $[a^\delta_n, a^\delta_{n+1})$. Therefore the insider's private information is fully, albeit gradually, revealed to the public so that the trading price does not jump when the fundamental value is announced. On the other hand, the total demand, in its own filtration, has the same distribution of the demand from noise traders, i.e., the insider is able to hide her trades among the noise trades. This is another manifestation of *inconspicuous trading theorem* commonly observed in the insider trading literature (cf. e.g., [32], [5], [7], etc.).

The insider's strategy discussed above is of feedback form. The insider can determine her trades only using the current total cumulative demand (and some additional randomness coming from the sequence of i.i.d. uniform distributed random variables which are also independent of the fundamental value and the noise trades). Even though this strategy is not optimal, its associated expected profit is close to the optimal value when the order size is small. Moreover the discrepancy converges to zero as the order size diminishes.

The Figure 2.1 presents a numeric example illustrates the convergence of the upper bound for insider's expected profit loss as the order size decreases to zero. In this example, $\tilde{v}$ takes values in $\{1, 2, 3\}$ with probability 0.55, 0.35, and 0.1, respectively. The expected profit in Kyle-Back equilibrium is 0.512. Compared to this, the following figure shows that the loss to insider's expected profit is small.

Finally, similar to Proposition 2.2.10 iii), insider's net order in the asymptotic Glosten-Milgrom equilibrium converges to the optimal strategy in the Kyle model as the order size decreases to zero.

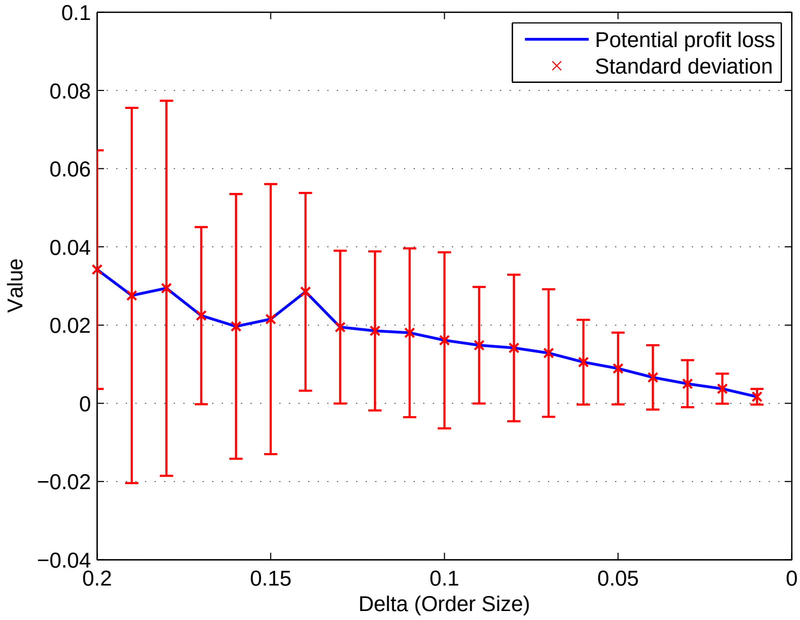


Figure 2.1: The mean and standard deviation of the upper bound for insider's expected profit loss. The figure is generated by Monte Carlo simulation with 10^5 paths.

Theorem 2.2.13 *Let $(X^{B,\delta}, X^{S,\delta}, \mathscr{F}^{I,\delta})_{\delta>0}$ be the sequence of insider's strategy in Theorem 2.2.12. Then*

$$X^{B,\delta} - X^{S,\delta} \xrightarrow{\mathcal{L}} X^0 \qquad \textit{as } \delta \downarrow 0,$$

where X^0 is the optimal strategy in Kyle model.

2.3 Optimizers in the HJB equation are not optimal control

Theorem 2.2.6 will be proved in this section. Let us first make the heuristic argument for the HJB equation rigorous by using the dynamic programming principle and standard arguments for viscosity solutions. To this end, recall the domain of Hamilton:

$$\begin{aligned}\mathrm{dom}(H) := \{(v_n, y, t, V) \in \{v_1, \cdots, v_N\} \times \delta\mathbb{Z} \times [0,1] \times \mathbb{R} \\ -\text{valued functions} \,|\, H(v_n, y, t, V) < \infty\}.\end{aligned}$$

Observe that control variables for (2.2.9) are chosen in $[0,\infty)$. Hence $(v_n, y, t, V) \in \mathrm{dom}(H)$ if

$$V(y+\delta, t) - V(y,t) + (v_n - p(y+\delta, t))\delta \leq 0, \tag{2.3.1}$$

$$V(y-\delta, t) - V(y,t) - (v_n - p(y-\delta, t))\delta \leq 0. \tag{2.3.2}$$

Moreover, when $(v_n, y, t, V) \in \mathrm{dom}(H)$, the Hamilton is reduced to

$$H(v_n, y, t, V) = (V(y+\delta, t) - 2V(y,t) + V(y-\delta, t))\beta. \tag{2.3.3}$$

Hence (2.2.8) reads

$$-V_t - (V(y+\delta,t) - 2V(y,t) + V(y-\delta,t))\beta = 0 \quad \text{in dom}(H). \tag{2.3.4}$$

Proposition 2.3.1 *The following statements hold for V^δ, $\delta > 0$:*

i) V^δ *is a viscosity solution of* (2.2.8)*;*

ii) $(v_n, y, t, V^\delta) \in$ *dom*(H) *for any* $n \in \{1, \cdots, N\}$ *and* $(y,t) \in \delta\mathbb{Z} \times [0,1)$*. Hence* V^δ *satisfies* (2.3.1)*,* (2.3.2)*, and is a viscosity solution of* (2.3.4)*;*

iii) $t \mapsto V^\delta(y,t)$ *is continuous on* $[0,1]$*;*

iv) $V^\delta(y,t) = \mathbb{E}^{\mathbb{P}^{\delta,y}}\left[V^\delta(Z_{s-t}, s)\right]$ *for any* $y \in \delta\mathbb{Z}$*, and* $0 \le t \le s \le 1$*.*

The proof is postponed to Appendix 2.7.1 where the dynamic programming principle together with the definition of viscosity solutions are recalled. The proof of Theorem 2.2.6 also requires the following result.

Lemma 2.3.2 *For any* $\delta > 0$*,* $n \in \{1, \cdots, N\}$*, and* $(y,t) \in \delta\mathbb{Z} \times [0,1)$*,* $V^\delta(v_n, y, t) > 0$*.*

Proof. Without loss of generality, we fix $\delta = 1$, $\tilde{v} = v_n$ for some $n \in \{1, \cdots, N\}$, and $(y,t) = (0,0)$. The superscript δ is omitted throughout this proof. When $n > 1$, let us construct a strategy where the insider buys once the asset is under-priced. Consider

$$\tau := \inf\{r \,:\, p(Z_{r-} + 1, r) < v_n)\} \wedge 1 \quad \text{and} \quad \sigma := \inf\{r > \tau \,:\, \Delta Y_r \neq 0\} \wedge 1.$$

Here τ is the first time that the asset is under-priced and σ is the arrival time of the first order after τ. The insider employs a strategy with intensity $\theta^{B,B}_r = \mathbb{I}_{\{\tau \le r \le \sigma\}}$ and all other intensities zero. Then the associated expected profit is

$$\mathbb{E}\left[\int_0^1 (v_n - a(Y_{r-}, r))\, \mathbb{I}_{\{\tau \le r \le \sigma\}} dr\right] = \mathbb{E}\left[\int_\tau^\sigma (v_n - p(Z_{r-} + 1, r))\, dr\right] > 0,$$

where the inequality follows from the definition of τ and the fact that $\mathbb{P}(\tau < 1) > 0$ due to Definition 2.2.1 ii). When $n = 1$, set $\tau := \inf\{t \,:\, p(Z_{t-} - 1, t) > v_1\} \wedge 1$ and $\theta^{S,S}_t = \mathbb{I}_{\{\tau \le t \le \sigma\}}$. Argument similar as above shows that this selling strategy also leads to positive expected profit. Therefore, in both cases, $V > 0$ is verified. $\square$

Proof of Theorem 2.2.6. Without loss of generality, we set $\delta = 1$ and omit the superscript δ throughout the proof.

Step 1: For any $n \in \{1, \cdots, N\}$, either one of the following situations holds:

- (2.3.1) holds as an equality and (2.3.2) is a strict inequality at all $(y,t) \in \mathbb{Z} \times [0,1)$;

- (2.3.2) holds as an equality and (2.3.1) is a strict inequality at all $(y, t) \in \mathbb{Z} \times [0, 1)$.

To prove the assertion, observe from (2.3.1) and (2.3.2) that

$$p(y,t) - v_n \leq V(y+1,t) - V(y,t) \leq p(y+1,t) - v_n, \quad (y,t) \in \mathbb{Z} \times [0,1).$$

Since $y \mapsto p(y,t)$ is strictly increasing for any $t \in [0,1)$, there exists $\eta(y,t) \in [0,1]$ such that

$$V(y+1,t) - V(y,t) = p(y,t) + \eta(y,t)\left(p(y+1,t) - p(y,t)\right) - v_n, \quad (y,t) \in \mathbb{Z} \times [0,1).$$

Assume that either (2.3.1) or (2.3.2) holds as an equality at some point. If such assumption fails, both inequalities in (2.3.1) and (2.3.2) are strict at all points in $\mathbb{Z} \times [0,1)$. Then all optimizers in (2.2.9) are identically zero, with the associated expected profit zero. Since $V > 0$, cf. Lemma 2.3.2, these trivial optimizers are not optimal strategies for (2.2.7). Hence the statement of the theorem is already confirmed in this trivial situation. Let us now assume (2.3.2) holds as an equality at (y_0+1, t_0), we will show (2.3.2) is an identity. On the other hand, combining the identity in (2.3.2) and the strict monotonicity of $y \mapsto p(y,t)$, we obtain

$$V(y+1,t) - V(y,t) = p(y,t) - v_n < p(y+1,t) - v_n, \quad (y,t) \in \mathbb{Z} \times [0,1),$$

hence the inequality (2.3.1) is always strict. The other situation where (2.3.1) is an identity and (2.3.2) is strict can be proved analogously.

Since (2.3.2) holds as an equality at (y_0+1, t_0), then, for any $s \in (t_0, 1)$,

$$\begin{aligned}\mathbb{E}^{y_0}\left[p(Z_{s-t_0}, s)\right] - v_n = p(y_0, t_0) - v_n &= V(y_0+1, t_0) - V(y_0, t_0) \\ &= \mathbb{E}^{y_0}\left[V(Z_{s-t_0}+1, s) - V(Z_{s-t_0}, s)\right],\end{aligned}$$

where the first identity follows from (2.2.5) and the Markov property of Z, the third identity is obtained after applying Proposition 2.3.1 iv) twice. On the other hand, the definition of $\eta(y,t)$ yields

$$\begin{aligned}&\mathbb{E}^{y_0}\left[V(Z_{s-t_0}+1, s) - V(Z_{s-t_0}, s)\right] \\ &= \mathbb{E}^{y_0}\left[p(Z_{s-t_0}, s) + \eta(Z_{s-t_0}, s)\left(p(Z_{s-t_0}+1, s) - p(Z_{s-t_0}, s)\right)\right] - v_n.\end{aligned}$$

The last two identities combined imply

$$\mathbb{E}^{y_0}\left[\eta(Z_{s-t_0}, s)\left(p(Z_{s-t_0}+1, s) - p(Z_{s-t_0}, s)\right)\right] = 0. \tag{2.3.5}$$

Recall that $\eta \geq 0$, $p(\cdot+1, s) - p(\cdot, s) > 0$ for any $s < 1$, and the distribution of Z_{s_0-t} has positive mass on each point in $\mathbb{Z}$. We then conclude from (2.3.5) that $\eta(y,s) = 0$ for any $y \in \mathbb{Z}$. Since s is arbitrarily chosen,

$$\eta(y,s) = 0, \quad \text{for any } y \in \mathbb{Z}, t_0 < s < 1. \tag{2.3.6}$$

Now fix s, the previous identity yields, for any $t < s$ and $y \in \mathbb{Z}$,

$$\begin{aligned} V(y+1,t) - V(y,t) &= \mathbb{E}^y\left[V(Z_{s-t}+1,s) - V(Z_{s-t},s)\right] \\ &= \mathbb{E}^y\left[p(Z_{s-t},s)\right] - v_n = p(y,t) - v_n, \end{aligned}$$

where Proposition 2.3.1 iv) is applied twice again to obtain the first identity. Therefore $\eta(y,t) = 0$ for any $y \in \mathbb{Z}$ and $t \leq s$, which combined with (2.3.6), implies (2.3.2) is an identity.

Step 2: Fix $1 < n < N$. When (2.3.2) is an identity, any optimizers in (2.2.9) are shown not to be the optimal strategy for (2.2.7). When (2.3.1) is an identity, a similar argument leads to the same conclusion. Combined with the result in Step 1, the statement of the theorem is confirmed.

When (2.3.2) is an identity, sending $t \to 1$, $V(y,1)$, defined as $\lim_{t\to 1} V(y,t)$, satisfies

$$V(y-1,1) - V(y,1) = v_n - P(y-1).$$

The previous identity and (2.2.6) combined imply that $V(y,1)$ is strictly decreasing when $y < a_n+1$, constant when $y \in [a_n+1, a_{n+1}+1)$, and strictly increasing when $y \geq a_{n+1}+1$. Thus $y \mapsto V(y,1)$ attains its minimum value when $y \in [a_n+1, a_{n+1}]$. Let $(\hat{X}^B, \hat{X}^S)$ be the point processes whose $\mathscr{F}^I$-intensities are optimizers $\hat{\theta}^{i,j}$, $i \in \{B,S\}$ and $j \in \{B,T,S\}$, in (2.2.9), and set $\hat{Y} = Z + \hat{X}^B - \hat{X}^S$. Assuming that $(\hat{X}^B, \hat{X}^S)$ is the optimal strategy for (2.2.7), DPP i) in Appendix 2.7.1 implies

$$\begin{aligned} V(y,t) \geq \mathbb{E}^{y,t}\Big[&V(\hat{Y}_1,1) \\ &+ \int_t^1 (v_n - p(\hat{Y}_{r-}+1,r))d\hat{X}_r^{B,B} + \int_t^1 (v_n - p(\hat{Y}_{r-}+2,r))d\hat{X}_r^{B,T} \\ &+ \int_t^1 (v_n - p(\hat{Y}_{r-},r))d\hat{X}_r^{B,S} - \int_t^1 (v_n - p(\hat{Y}_{r-}-1,r))d\hat{X}_r^{S,S} \\ &- \int_t^1 (v_n - p(\hat{Y}_{r-}-2,r))d\hat{X}_r^{S,T} - \int_t^1 (v_n - p(\hat{Y}_{r-},r))d\hat{X}_r^{S,B}\Big], \end{aligned}$$

where the expectation is taken under $\mathbb{P}^{y,t}$ with $\mathbb{P}^{y,t}(\hat{Y}_t = y) = 1$. However, the value function $V(y,t)$ is exactly the expected profit when the insider employs the optimal strategy $(\hat{X}^B, \hat{X}^S)$. Therefore, the previous identity yields

$$\mathbb{E}^{y,t}[V(\hat{Y}_1,1)] = 0.$$

Recall that $V(\cdot,1)$, as limit of positive functions, is nonnegative, and it achieves the minimum at $[a_n+1, a_{n+1}]$. The previous identity implies $V(y,1) = 0$ when $y \in [a_n+1, a_{n+1}]$ and

$$\hat{Y}_1 \in [a_n+1, a_{n+1}], \qquad \mathbb{P}^{y,t} - a.s.. \tag{2.3.7}$$

However, when (2.3.2) is an identity and (2.3.1) is a strict inequality, any optimizer of (2.2.9) satisfies $\hat{\theta}^{B,B} = \hat{\theta}^{B,S} \equiv 0$, i.e., $\hat{X}^B \equiv 0$. Therefore, $\hat{Y} = Z^B - Z^S - \hat{X}^S$ with only negative controlled jumps from $\hat{X}^S$ cannot compensate Z^S to satisfy (2.3.7), where $[a_n+1, a_{n+1}]$ is a finite interval in $\mathbb{Z}$ when $1 < n < N$. □

2.4 A suboptimal strategy

We start to prepare the proof of Theorem 2.2.12 from this section.

For the rest of the paper, $N < \infty$, assumed in Theorem 2.2.12, is enforced unless stated otherwise.

In this section we are going to characterize a suboptimal strategy of feedback form in the Glosten-Milgrom model with order size δ, such that the pricing rule (2.2.5) is rational. To simplify presentation, we will take $\delta = 1$, hence omit all superscript δ, throughout this section. Scaling all processes by δ gives the desired processes when the order size is δ.

The following standing assumption on distribution of $\tilde{v}$ will be enforced throughout this section:

Assumption 2.4.1 There exists a strictly increasing sequence $(a_n)_{n=1,\cdots,N+1}$ such that

i) $a_n \in \mathbb{Z} \cup \{-\infty, \infty\}$, $a_1 = -\infty$, $a_{N+1} = \infty$, and $\cup_{n=1}^{N}[a_n, a_{n+1}) = \mathbb{Z} \cup \{-\infty\}$;

ii) $\mathbb{P}(Z_1 \in [a_n, a_{n+1})) = \mathbb{P}(\tilde{v} = v_n)$, $n = 1, \cdots, N$;

iii) The middle level $m_n = (a_n + a_{n+1} - 1)/2$ of the interval $[a_n, a_{n+1})$ is not an integer.

Item i) and ii) have already been assumed in Assumption 2.2.4. Item iii) is a technical assumption which facilities the construction of the suboptimal strategy. In the next section, when an arbitrary $\tilde{v}$ of distribution (2.1.1) is considered and the order size δ converges to zero, a sequence $(a_n^\delta)_{n=1,\cdots,N+1,\delta>0}$ together with a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$ will be constructed, such that Assumption 2.4.1 is satisfied for each δ and $\tilde{v}^\delta$ converges to $\tilde{v}$ in law. To simplify notation, we denote by $\underline{m}_n := \lfloor (a_n + a_{n+1} - 1)/2 \rfloor$ the largest integer smaller than m_n and by $\overline{m}_n := \lceil (a_n + a_{n+1} - 1)/2 \rceil$ the smallest integer larger than m_n. Assumption 2.4.1 iii) implies $a_n \leq \underline{m}_n < m_n < \overline{m}_n < a_{n+1}$ and $\overline{m}_n - \underline{m}_n = 1$ when both a_n and a_{n+1} are finite.

Let us now define a function U, which relates to the expected profit of a suboptimal strategy and also dominates the value function V. First the Markov property Z implies that p is continuously differentiable in the time variable and satisfies[8]

$$\begin{aligned} &p_t + (p(y+1,t) - 2p(y,t) + p(y-1,t))\,\beta = 0, \quad (y,t) \in \mathbb{Z} \times [0,1), \\ &p(y,1) = P(y). \end{aligned} \tag{2.4.1}$$

Define

$$U(v_n, y, 1) := \sum_{j=y}^{a_n - 1} (v_n - A(j))\, \mathbb{I}_{\{y \leq \underline{m}_n\}} + \sum_{j=a_{n+1}}^{y} (B(j) - v_n)\, \mathbb{I}_{\{y \geq \overline{m}_n\}}, \quad y \in \mathbb{Z}, 1 \leq n \leq N, \tag{2.4.2}$$

[8] This follows from the same argument as in [18, Footnote 4].

where $A(y) := P(y+1)$ and $B(y) := P(y-1)$ can be considered as ask and bid pricing functions right before time 1. Since $(v_n)_{n=1,\cdots,N}$ is increasing, $U(\cdot,\cdot,1)$ is non-negative and

$$U(v_n, y, 1) = 0 \quad \Longleftrightarrow \quad y \in [a_n - 1, a_{n+1} + 1). \tag{2.4.3}$$

Given $U(\cdot,\cdot,1)$ as above, U is extended to $t \in [0,1)$ as follows:

$$U(v_n, y, t) := U(v_n, y, 1) + \beta \int_t^1 (p(y,r) - p(y-1,r))\, dr, \quad y \geq \overline{m}_n, \tag{2.4.4}$$

$$U(v_n, y, t) := U(v_n, y, 1) + \beta \int_t^1 (p(y+1,r) - p(y,r))\, dr, \quad y \leq \underline{m}_n, \tag{2.4.5}$$

for $t \in [0,1)$ and $n = 1, \cdots, N$. Since N is finite, p is bounded, hence U takes finite value.

Proposition 2.4.2 *Let Assumption 2.4.1 hold. Suppose that the market maker chooses p in* (2.2.5) *as the pricing rule. Then for any insider's admissible strategy* $(X^B, X^S; \mathscr{F}^I)$, *with* $\mathscr{F}^I$*-intensities* $\theta^{i,j}, i \in \{B,S\}$ *and* $j \in \{B,T,S\}$, *the associated expected profit function* $\mathcal{J}(v_n, y, t; X^B, X^S)$ *satisfies*

$$\mathcal{J}(v_n, y, t; X^B, X^S) \leq U(v_n, y, t) - L(v_n, y, t), \quad n \in \{1, \cdots, N\}, (y,t) \in \mathbb{Z} \times [0,1]. \tag{2.4.6}$$

where

$$\begin{aligned} L(v_n, y, t) := &\mathbb{E}^y\Bigg[\int_t^1 (v_n - p(\underline{m}_n, r))\left[\left(\beta - \theta_r^{B,S} + \theta_r^{S,S}\right) \mathbb{I}_{\{Y_{r-} = \overline{m}_n\}}\right. \\ &\qquad \left. + \theta_r^{S,T}\, \mathbb{I}_{\{Y_{r-} = \overline{m}_n + 1\}}\right] dr \bigg| \tilde{v} = v_n\Bigg] \\ &- \mathbb{E}^y\Bigg[\int_t^1 (v_n - p(\overline{m}_n, r))\left[\left(\beta - \theta_r^{S,B} + \theta_r^{B,B}\right) \mathbb{I}_{\{Y_{r-} = \underline{m}_n\}}\right. \\ &\qquad \left. + \theta_r^{B,T}\, \mathbb{I}_{\{Y_{r-} = \underline{m}_n - 1\}}\right] dr \bigg| \tilde{v} = v_n\Bigg]. \end{aligned} \tag{2.4.7}$$

Moreover (2.4.6) *is an identity when the following conditions are satisfied:*

i) $Y_1 \in [a_n - 1, a_{n+1} + 1)$ *a.s. when* $\tilde{v} = v_n$;

ii) $X_t^{S,S} = X_t^{S,B} \equiv 0$ *when* $Y_{t-} \leq \underline{m}_n$, $X_t^{B,B} = X_t^{B,S} \equiv 0$ *when* $Y_{t-} \geq \overline{m}_n$, $\theta^{B,T} \equiv 0$ *when* $y \geq \underline{m}_n$, *and* $\theta^{S,T} \equiv 0$ *when* $y \leq \overline{m}_n$.

Before proving this result, let us derive equations that U satisfies. The following result shows that U satisfies (2.3.4) except when $y = \overline{m}_n$ and $y = \underline{m}_n$, and U satisfies the identity in either (2.3.1) or (2.3.2) depending on whether $y \leq \underline{m}_n$ or $y \geq \overline{m}_n$.

Lemma 2.4.3 *The function U satisfies the following equations: (Here* $\tilde{v} = v_n$ *is fixed and the dependence on* $\tilde{v}$ *is omitted in U.)*

$$U_t + (U(y+1,t) - 2U(y,t) + U(y-1,t))\,\beta = 0, \quad y > \overline{m}_n \text{ or } y < \underline{m}_n, \tag{2.4.8}$$

$$U_t + (U(y+1,t) - 2U(y,t) + U(y-1,t))\,\beta = (p(\underline{m}_n, t) - v_n)\beta, \quad y = \overline{m}_n, \tag{2.4.9}$$

$$U_t + (U(y+1,t) - 2U(y,t) + U(y-1,t))\,\beta = (v_n - p(\overline{m}_n, t))\beta, \quad y = \underline{m}_n, \tag{2.4.10}$$

$$U(y,t) - U(y+1,t) - (v_n - p(y,t)) = 0, \quad y \geq \overline{m}_n, \tag{2.4.11}$$

$$U(y,t) - U(y-1,t) + (v_n - p(y,t)) = 0, \quad y \leq \underline{m}_n. \tag{2.4.12}$$

Proof. We will only verify these equations when $y \geq \overline{m}_n$. The remaining equations can be proved similarly. First (2.4.2) implies

$$U(y+1,1) - U(y,1) = B(y+1) - v_n = P(y) - v_n, \qquad y \geq \overline{m}_n.$$

Combining the previous identity with (2.4.4),

$$\begin{aligned} U(y+1,t) - U(y,t) &= U(y+1,1) - U(y,1) \\ &\quad + \beta \int_t^1 (p(y+1,r) - 2p(y,r) + p(y-1,r))\, dr \\ &= p(y,t) - v_n, \end{aligned}$$

where (2.4.1) is used to obtain the second identity. This verifies (2.4.11). When $y > \overline{m}_n$, summing up (2.4.11) at y and $y+1$, and taking time derivative in (2.4.4), yield

$$\begin{aligned} &U_t + (U(y+1,t) - 2U(y,t) + U(y-1,t))\,\beta \\ &= -\beta(p(y,t) - p(y-1,t)) + \beta(p(y,t) - p(y-1,t)) \\ &= 0, \end{aligned}$$

which confirms (2.4.8) when $y > \overline{m}_n$. When $y = \overline{m}_n$, observe from (2.4.2), (2.4.4) and (2.4.5) that $U(\overline{m}_n, \cdot) = U(\underline{m}_n, \cdot)$. Then

$$\begin{aligned} &U_t + (U(y+1,t) - 2U(y,t) + U(y-1,t))\,\beta \\ &= -\beta\,(p(\overline{m}_n,t) - p(\underline{m}_n,t)) + \beta\,(U(\overline{m}_n+1,t) - U(\overline{m}_n,t)) \\ &= -\beta\,(p(\overline{m}_n,t) - p(\underline{m}_n,t)) - \beta\,(v_n - p(\overline{m}_n,t)) \\ &= \beta\,(p(\underline{m}_n,t) - v_n)\,, \end{aligned}$$

where the second identity follows from (2.4.11). □

Proof of Proposition 2.4.2. Throughout the proof the $\tilde{v} = v_n$ is fixed and the dependence on $\tilde{v}$ is omitted in U. Let $Y^B = Z^B + X^{B,B} + X^{B,T} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} + X^{S,T} - X^{B,S}$ be positive and negative parts of Y respectively. Then $Y^B - \int_0^\cdot (\beta - \theta_r^{S,B} - \theta_r^{B,T}) dr - \int_0^\cdot \theta_r^{B,B} dr - 2\int_0^\cdot \theta_r^{B,T} dr$ and $Y^S - \int_0^\cdot (\beta - \theta_r^{B,S} - \theta_r^{S,T}) dr - \int_0^\cdot \theta_r^{S,S} dr - 2\int_0^\cdot \theta_r^{S,T} dr$ are $\mathscr{F}^I$-martingales. Applying Itô's formula to $U(Y_\cdot, \cdot)$, we obtain

$$\begin{aligned}
&U(Y_1, 1)\\
&= U(y,t) + \int_t^1 U_t(Y_{r-}, r)dr\\
&\quad + \int_t^1 \left[U(Y_r, r) - U(Y_{r-}, r)\right] dY_r^B + \int_t^1 \left[U(Y_r, r) - U(Y_{r-}, r)\right] dY_r^S\\
&= U(y,t) + \int_t^1 \left[U_t(Y_{r-}, r) + \left(U(Y_{r-}+1, r) - 2U(Y_{r-}, r) + U(Y_{r-}-1, r)\right)\beta\right] dr\\
&\quad + \int_t^1 \left[U(Y_{r-}+1, r) - U(Y_{r-}, r)\right] \left(\theta_r^{B,B} - \theta_r^{S,B}\right) dr\\
&\quad + \int_t^1 \left[U(Y_{r-}+2, r) - U(Y_{r-}+1, r)\right] \theta_r^{B,T} dr\\
&\quad + \int_t^1 \left[U(Y_{r-}-1, r) - U(Y_{r-}, r)\right] \left(\theta_r^{S,S} - \theta_r^{B,S}\right) dr\\
&\quad + \int_t^1 \left[U(Y_{r-}-2, r) - U(Y_{r-}-1, r)\right] \theta_r^{S,T} dr + M_1 - M_t,
\end{aligned} \tag{2.4.13}$$

where

$$\begin{aligned}
M = &\int_0^\cdot \left[U(Y_r, r) - U(Y_{r-}, r)\right] d\left(Y_r^B - \int_0^r \left(\beta - \theta_u^{S,B} + \theta_u^{B,B} + \theta_u^{B,T}\right) du\right)\\
&+ \int_0^\cdot \left[U(Y_r, r) - U(Y_{r-}, r)\right] d\left(Y_r^S - \int_0^r \left(\beta - \theta_u^{B,S} + \theta_u^{S,S} + \theta_u^{S,T}\right) du\right).
\end{aligned}$$

Since (2.4.11) and (2.4.12) imply $U(y+1,t) - U(y,t)$ is either $p(y,t) - v_n$ or $p(y+1,t) - v_n$, which are both bounded from below by $v_1 - v_n$ and from above by $v_N - v_n$, hence M is an $\mathscr{F}^I$-martingale (cf. [13, Chapter I, T6]). On the right hand side of (2.4.13), splitting the second integral on $\{Y_{r-} \geq \overline{m}_n\}$, $\{Y_{r-} = \underline{m}_n\}$, and $\{Y_{r-} < \underline{m}_n\}$, splitting the fourth integral on $\{Y_{r-} > \overline{m}_n\}$, $\{Y_{r-} = \overline{m}_n\}$, and $\{Y_{r-} \leq \underline{m}_n\}$, utilizing $U(\overline{m}_n, \cdot) = U(\underline{m}_n, \cdot)$, as well as different equations in Lemma 2.4.3 in different regions, we obtain

$$\begin{aligned}
&U(Y_1, 1)\\
&= U(y,t) + \int_t^1 \left(p(\underline{m}_n, r) - v_n\right) \beta \mathbb{I}_{\{Y_{r-} = \overline{m}_n\}} dr + \int_t^1 \left(v_n - p(\overline{m}_n, r)\right) \beta \mathbb{I}_{\{Y_{r-} = \underline{m}_n\}} dr\\
&\quad - \int_t^1 \left(v_n - p(Y_{r-}, r)\right) \mathbb{I}_{\{Y_{r-} \geq \overline{m}_n\}} (\theta_r^{B,B} - \theta_r^{S,B}) dr\\
&\quad - \int_t^1 \left(v_n - p(Y_{r-}+1, r)\right) \mathbb{I}_{\{Y_{r-} < \underline{m}_n\}} (\theta_r^{B,B} - \theta_r^{S,B}) dr\\
&\quad - \int_t^1 \left(v_n - p(Y_{r-}+1, r)\right) \mathbb{I}_{\{Y_{r-} \geq \underline{m}_n\}} \theta_r^{B,T} dr\\
&\quad - \int_t^1 \left(v_n - p(Y_{r-}+2, r)\right) \mathbb{I}_{\{Y_{r-} < \underline{m}_n - 1\}} \theta_r^{B,T} dr\\
&\quad + \int_t^1 \left(v_n - p(Y_{r-}-1, r)\right) \mathbb{I}_{\{Y_{r-} > \overline{m}_n\}} (\theta_r^{S,S} - \theta_r^{B,S}) dr\\
&\quad + \int_t^1 \left(v_n - p(Y_{r-}, r)\right) \mathbb{I}_{\{Y_{r-} \leq \underline{m}_n\}} (\theta_r^{S,S} - \theta_r^{B,S}) dr
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^1 (v_n - p(Y_{r-}-2,r))\,\mathbb{I}_{\{Y_{r-}>\overline{m}_n+1\}}\theta_r^{S,T}dr \\
&+\int_t^1 (v_n - p(Y_{r-}-1,r))\,\mathbb{I}_{\{Y_{r-}\leq\overline{m}_n\}}\theta_r^{S,T}dr + M_1 - M_t.
\end{aligned}
$$

Rearranging the previous identity by putting the profit of (X^B, X^S) to the left hand side, we obtain

$$
\begin{aligned}
&U(y,t) - U(Y_1,1) - K - L + M_1 - M_t \\
&= \int_t^1 (v_n - p(Y_{r-}+1,r))\theta_r^{B,B}dr + \int_t^1 (v_n - p(Y_{r-}+2,r))\,\theta_r^{B,T}dr \\
&\quad + \int_t^1 (v_n - p(Y_{r-},r))\theta_r^{B,S}dr - \int_t^1 (v_n - p(Y_{r-}-1,r))\theta_r^{S,S}dr \\
&\quad - \int_t^1 (v_n - p(Y_{r-}-2,r))\,\theta_r^{S,T}dr - \int_t^1 (v_n - p(Y_{r-},r))\theta_r^{S,B}dr
\end{aligned}
\tag{2.4.14}
$$

where

$$
\begin{aligned}
K = &\int_t^1 (p(Y_{r-}+1,r) - p(Y_{r-},r))\,\mathbb{I}_{\{Y_{r-}\geq\overline{m}_n\}}\theta_r^{B,B}dr \\
&+\int_t^1 (p(Y_{r-},r) - p(Y_{r-}-1,r))\,\mathbb{I}_{\{Y_{r-}\geq\overline{m}_n\}}\theta_r^{B,S}dr \\
&+\int_t^1 (p(Y_{r-}+2,r) - p(Y_{r-}+1,r))\,\mathbb{I}_{\{Y_{r-}\geq\underline{m}_n\}}\theta_r^{B,T}dr \\
&+\int_t^1 (p(Y_{r-},r) - p(Y_{r-}-1,r))\,\mathbb{I}_{\{Y_{r-}\leq\underline{m}_n\}}\theta_r^{S,S}dr \\
&+\int_t^1 (p(Y_{r-}+1,u) - p(Y_{r-},r))\,\mathbb{I}_{\{Y_{r-}\leq\underline{m}_n\}}\theta_r^{S,B}dr \\
&+\int_t^1 (p(Y_{r-}-1,r) - p(Y_{r-}-2,r))\,\mathbb{I}_{\{Y_{r-}\leq\overline{m}_n\}}\theta_r^{S,T}dr, \\
L = &\int_t^1 [v_n - p(\underline{m}_n,r)]\left[(\beta - \theta_r^{B,S} + \theta_r^{S,S})\mathbb{I}_{\{Y_{r-}=\overline{m}_n\}} + \theta_r^{S,T}\mathbb{I}_{\{Y_{r-}=\overline{m}_n+1\}}\right]dr \\
&-\int_t^1 [v_n - p(\overline{m}_n,r)]\left[(\beta - \theta_r^{S,B} + \theta_r^{B,B})\mathbb{I}_{\{Y_{r-}=\underline{m}_n\}} + \theta_r^{B,T}\mathbb{I}_{\{Y_{r-}=\underline{m}_n-1\}}\right]dr.
\end{aligned}
$$

Taking conditional expectation $\mathbb{E}[\cdot|\mathscr{F}_t^I, Y_t = y]$ on both sides of (2.4.14), the right hand side is the expected profit $\mathcal{J}(X^B, X^S)$, while, on the left hand side, both $U(\cdot, 1)$ and K are non-negative (cf. Definition 2.2.1 i)). Therefore (2.4.6) is verified. To attain the identity in (2.4.6), we need i) $Y_1 \in [a_n - 1, a_{n+1} + 1)$ a.s. so that $U(Y_1, 1) = 0$ a.s. follows from (2.4.3); ii) $\theta^{B,B} = \theta^{B,S} \equiv 0$ when $y \geq \overline{m}_n$, $\theta^{S,S} = \theta^{S,B} \equiv 0$ when $y \leq \underline{m}_n$, $\theta^{B,T} \equiv 0$ when $y \geq \underline{m}_n$, and $\theta^{S,T} \equiv 0$ when $y \leq \overline{m}_n$. □

Come back to the statement of Proposition 2.4.2. If the insider chooses a strategy such that both conditions in i) and ii) are satisfied, then the identity in (2.4.6) is attained, hence the expected profit of this strategy is $U - L$. On the other hand, define $U^S : \{v_1, \cdots, v_N\} \times \mathbb{Z} \times [0,1] \to \mathbb{R}$ via

$$
U^S(v_n, y, t) = \begin{cases} U(v_n, y, t) & y \geq \overline{m}_n \\ U(v_n, y-1, t) & y \leq \underline{m}_n \end{cases}. \tag{2.4.15}
$$

The next result shows that U^S dominates the value function V, therefore $U^S - U + L$ is the upper bound of the potential loss of the expected profit. In section 2.6, we will prove this potential loss converges to zero as $\delta \downarrow 0$. Therefore, when the order size is small, the insider losses little expected profit by employing a strategy satisfying Proposition 2.4.2 i) and ii).

Proposition 2.4.4 *Let Assumption 2.4.1 hold. Then $V \leq U^S$, hence $V < \infty$, on $\{v_1, \cdots, v_N\} \times \mathbb{Z} \times [0,1]$.*

Proof. Fix v_n and omit it as the first argument of U^S and U throughout the proof. We first verify

$$U^S(y,t) - U^S(y+1,t) - (v_n - p(y,t)) = 0, \tag{2.4.16}$$

$$U^S_t + \left(U^S(y+1,t) - 2U^S(y,t) + U^S(y-1,t)\right)\beta = 0, \tag{2.4.17}$$

for any $(y,t) \in \mathbb{Z} \times [0,1)$. Indeed, when $y \geq \overline{m}_n$, (2.4.16) is exactly (2.4.11). When $y = \underline{m}_n$,

$$\begin{aligned} U^S(\underline{m}_n, t) - U^S(\overline{m}_n, t) &= U(\underline{m}_n - 1, t) - U(\overline{m}_n, t) \\ &= U(\underline{m}_n - 1, t) - U(\underline{m}_n, t) = v_n - p(\underline{m}_n, t), \end{aligned}$$

where the second identity follows from $U(\overline{m}_n, t) = U(\underline{m}_n, t)$ and the third identity holds due to (2.4.12). When $y < \underline{m}_n$,

$$U^S(y,t) - U^S(y+1,t) = U(y-1,t) - U(y,t) = v_n - p(y,t),$$

where (2.4.12) is utilized again to obtain the second identity. Therefore (2.4.16) is confirmed for all cases. As for (2.4.17), (2.4.16) yields

$$U^S(y+1,t) - 2U^S(y,t) + U^S(y-1,t) = p(y,t) - p(y-1,t).$$

On the other hand, we have from (2.4.4) and (2.4.5) that

$$U^S_t(y,t) = \begin{cases} U_t(y,t) = -\beta(p(y,t) - p(y-1,t)) & y \geq \overline{m}_n \\ U_t(y-1,t) = -\beta(p(y,t) - p(y-1,t)) & y \leq \underline{m}_n \end{cases}.$$

Therefore (2.4.17) is confirmed after combining the previous two identities.

Now note that $U^S(\cdot, 1) \geq 0$, moreover U^S satisfies (2.4.16) and (2.4.17). The assertion $V \leq U^S$ follows from the same argument as in the high type of [18, Proposition 3.2]. □

Having studied the insider's optimization problem, let us turn to the market maker. Given $(X^B, X^S; \mathscr{F}^I)$, Definition 2.2.11 ii) requires the pricing rule to be rational. This leads to another constraint on $(X^B, X^S; \mathscr{F}^I)$.

Proposition 2.4.5 *If there exists an admissible strategy $(X^B, X^S; \mathscr{F}^I)$ such that* *i)* $Y^B = Z^B + X^{B,B} + X^{B,T} - X^{S,B}$ *and* $Y^S = Z^S + X^{S,S} + X^{S,T} - X^{B,S}$ *are independent* $\mathscr{F}^Y-$*adapted Poisson processes with common intensity* β*;*

ii) $[Y_1 \in [a_n, a_{n+1})] = [\tilde{v} = v_n]$*,* $n = 1, \cdots, N$*.*

Then the pricing rule (2.2.5) *is rational.*

Proof. For any $t \in [0, 1]$,

$$p(Y_t, t) = \mathbb{E}^{Y_t}[P(Z_{1-t})] = \mathbb{E}\left[P(Z_1) \,|\, Z_t = Y_t\right] = \mathbb{E}\left[P(Y_1) \,|\, \mathscr{F}_t^Y\right] = \mathbb{E}[\tilde{v} \,|\, \mathscr{F}_t^Y],$$

where the third identity holds since Y and Z have the same distribution, the fourth identity follows from ii) and (2.2.6). □

Remark 2.4.6 If the insider places a buy (resp. sell) order when a noise buy (resp. sell) order arrives, Proposition 2.4.5 i) cannot be satisfied. Therefore in the asymptotic equilibrium the insider will not trade in the same direction as the noise traders, i.e., $X^{B,T} = X^{S,T} \equiv 0$, so that the market maker can employ a rational pricing rule.

Concluding this section, we need to construct point processes $(X^B, X^S; \mathscr{F}^I)$ which simultaneously satisfy conditions in Proposition 2.4.2 ii), Proposition 2.4.5 i) and ii)[9]. This construction is a natural extension of [18, Section 4], where $N = 2$ is considered, and will be presented in the next section.

2.5 Construction of a point process bridge

In this section, we will construct point processes X^B and X^S on a probability space $(\Omega, \mathscr{F}^I, (\mathscr{F}_t^I)_{t\in[0,1]}, \mathbb{P})$ such that $X^{B,T} = X^{S,T} \equiv 0$, due to Remark 2.4.6, and satisfy

i) $Y^B = Z^B + X^{B,B} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} - X^{B,S}$ are independent $\mathscr{F}^Y$-adapted Poisson processes with common intensity β;

ii) $X_t^{B,B} = X_t^{B,S} \equiv 0$ when $Y_{t-} \geq \overline{m}_n$, $X_t^{S,S} = X_t^{S,B} \equiv 0$ when $Y_{t-} \leq \underline{m}_n$;

iii) $[Y_1 \in [a_n, a_{n+1})] = [\tilde{v} = v_n]$ $\mathbb{P}$-a.s. for $n = 1, \cdots, N$.

The construction is a natural extension of [18] where $N = 2$ is considered. As in [18], X^B and X^S are constructed using two independent sequences of iid random variables $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ with uniform distribution on $[0, 1]$, moreover they are independent of Z and $\tilde{v}$. The insider uses $(\eta_i)_{i\geq 1}$ to randomly contribute either buy or sell orders, and uses $(\zeta_i)_{i\geq 1}$ to randomly cancel noise orders.

[9]Note is Proposition 2.4.5 ii) implies Proposition 2.4.2 i).

Throughout this section Assumption 2.4.1 is enforced. Moreover, we set $\delta = 1$, hence suppress the superscript δ. Otherwise X^B and X^S can be scaled by δ to obtain the desired processes.

In the following construction, we will define a probability space $(\Omega, \mathscr{F}^I, (\mathscr{F}^I_t)_{t\in[0,1]}, \mathbb{P})$ on which Y takes the form

$$Y = Z + \sum_{n=1}^{N} \mathbb{I}_{\{A_n\}}(X^B - X^S). \tag{2.5.1}$$

Here Z is the difference of two independent $\mathscr{F}^I$-adapted Poisson processes with intensity β, $A_n \in \mathscr{F}^I_0$ such that $\mathbb{P}(A_n) = \mathbb{P}(Z_1 \in [a_n, a_{n+1}))$ for each $n = 1, \cdots, N$.

Before constructing X^B and X^S satisfying desired properties, let us draw some intuition from the theory of filtration enlargement. Let us define $(\mathbb{D}([0,1],\mathbb{Z}), \overline{\mathscr{F}}, (\overline{\mathscr{F}}_t)_{t\in[0,1]}, \overline{\mathbb{P}})$ be the canonical space where $\mathbb{D}([0,1],\mathbb{Z})$ is $\mathbb{Z}$-valued càdlàg functions, $\overline{\mathbb{P}}$ is a probability measure under which Z^B and Z^S are independent Poisson processes with intensities β, $(\overline{\mathscr{F}}_t)_{t\in[0,1]}$ is the minimal filtration generated by Z^B and Z^S satisfying the usual conditions, and $\overline{\mathscr{F}} = \vee_{t\in[0,1]}\overline{\mathscr{F}}_t$. Let us denote by $(\mathcal{G}_t)_{t\in[0,1]}$ the filtration $(\overline{\mathscr{F}}_t)_{t\in[0,1]}$ enlarged with a sequence of random variables $(\mathbb{I}_{\{Z_1\in[a_n,a_{n+1})\}})_{n=1,\cdots,N}$.

In order to find the $\mathcal{G}$-intensities of Z^B and Z^S, we use a standard enlargement of filtration argument which can be found, e.g. in [35]. To this end, recall $h_n(y,t) = \overline{\mathbb{P}}[Z_1 \in [a_n, a_{n+1}) \,|\, Z_t = y]$. Note that h_n is strictly positive on $\mathbb{Z} \times [0,1)$. Moreover the Markov property of Z implies h_n is continuously differentiable in the time variable and satisfies

$$\begin{aligned} &\partial_t h_n + (h_n(y+1,t) - 2h_n(y,t) + h_n(y-1,t))\,\beta = 0, \quad (y,t) \in \mathbb{Z}\times[0,1), \\ &h_n(y,1) = \mathbb{I}_{\{y\in[a_n,a_{n+1})\}}. \end{aligned} \tag{2.5.2}$$

Lemma 2.5.1 *The $\mathcal{G}$-intensities of Z^B and Z^S at $t \in [0,1)$ are given by*

$$\sum_{n=1}^{N} \mathbb{I}_{\{Z_1\in[a_n,a_{n+1})\}} \frac{h_n(Z_{t-}+1,t)}{h_n(Z_{t-},t)}\beta \quad \textit{and} \quad \sum_{n=1}^{N} \mathbb{I}_{\{Z_1\in[a_n,a_{n+1})\}} \frac{h_n(Z_{t-}-1,t)}{h_n(Z_{t-},t)}\beta,$$

respectively.

Proof. We will only calculate the intensity for Z^B. The intensity of Z^S can be obtained similarly. All expectations are taken under $\overline{\mathbb{P}}$ throughout this proof. For $s \le t < 1$, take an arbitrary $E \in \overline{\mathscr{F}}_s$ and denote $M^B_t := Z^B_t - \beta t$. The definition of h_n and the $\overline{\mathscr{F}}$-martingale property of M^B imply

$$\begin{aligned} &\mathbb{E}\left[(M^B_t - M^B_s)\mathbb{I}_{\{E\}}\mathbb{I}_{\{Z_1\in[a_n,a_{n+1})\}}\right] \\ &= \mathbb{E}\left[(M^B_t - M^B_s)\mathbb{I}_{\{E\}} h_n(Z_t,t)\right] \\ &= \mathbb{E}\left[\mathbb{I}_{\{E\}}(\langle M^B, h_n(Z_\cdot,\cdot)\rangle_t - \langle M^B, h_n(Z_\cdot,\cdot)\rangle_s)\right] \\ &= \mathbb{E}\left[\mathbb{I}_{\{E\}} \int_s^t \beta\left(h_n(Z_{r-}+1,r) - h_n(Z_{r-},r)\right) dr\right] \\ &= \mathbb{E}\left[\mathbb{I}_{\{E\}} \int_s^t \beta\, \mathbb{I}_{\{Z_1\in[a_n,a_{n+1})\}} \frac{h_n(Z_{r-}+1,r) - h_n(Z_{r-},r)}{h_n(Z_{r-},r)} dr\right]. \end{aligned}$$

These computations for each $n = 1, \cdots, N$ imply that

$$M^B - \int_s^{\cdot} \beta \sum_{n=1}^{N} \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}} \frac{h_n(Z_{r-}+1, r) - h_n(Z_{r-}, r)}{h_n(Z_{r-}, r)} dr$$

defines a $\mathcal{G}$-martingale. Therefore the $\mathcal{G}$-intensity of Z^B follows from $Z_t^B = M_t^B + \beta t$. □

To better understand intensities in the previous lemma, let us collect several properties for h_n:

Lemma 2.5.2 *Let Assumption 2.4.1 hold. The following properties hold for each h_n, $n = 1, \cdots, N$:*

i) $h_n(\cdot, \cdot) = h_n(2m_n - \cdot, \cdot)$; in particular, $h_n(\overline{m}_n, \cdot) = h_n(\underline{m}_n, \cdot)$.

ii) $y \mapsto h_n(y, t)$ is strictly increasing when $y \le \underline{m}_n$ and strictly decreasing when $y \ge \overline{m}_n$.

Here, when $n = 1$ (resp. $n = N$), $\overline{m}_n = \underline{m}_n = -\infty$ (resp. $\overline{m}_n = \underline{m}_n = \infty$).

Proof. Recall that $a_n + a_{n+1} - 1 = 2m_n$. Then

$$\begin{aligned}
h_n(y,t) &= \overline{\mathbb{P}}[Z_1 \in [a_n, a_{n+1}) \,|\, Z_t = y] = \overline{\mathbb{P}}[y + Z_{1-t} \in [a_n, a_{n+1})] \\
&= \overline{\mathbb{P}}[2m_n - y - Z_{1-t} \in (2m_n - a_{n+1}, 2m_n - a_n]] \\
&= \overline{\mathbb{P}}[2m_n - y - Z_{1-t} \in [a_n, a_{n+1})] = h_n(2m_n - y, t),
\end{aligned}$$

where the last identity holds since Z and $-Z$ have the same distribution. This verifies i). To prove ii), rewrite $h_n(y,t) = \overline{\mathbb{P}}[Z_{1-t} \in [a_n - y, a_{n+1} - y)]$. Then the statement ii) follows from the fact that $y \mapsto \mathbb{P}(Z_{1-t} = y)$ is strictly increasing when $y \le 0$ and strictly decreasing when $y \ge 0$. □

In what follows, given $A_n \in \mathscr{F}_0^I$ such that $\mathbb{P}(A_n) = \overline{\mathbb{P}}(Z_1 \in [a_n, a_{n+1}))$, $(X^B, X^S; \mathscr{F}^I)$ on A_n will be constructed so that $\mathscr{F}^I$-intensity of Y^B (resp. Y^S) on A_n match $\mathcal{G}$-intensities of Z^B (resp. Z^S) on $[Z_1 \in [a_n, a_{n+1})]$. Matching these intensities ensures that $(X^B, X^S; \mathscr{F}^I)$ satisfies desired properties, cf. Proposition 2.5.5 below. Recall $Y^B = Z^B + X^{B,B} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} - X^{B,S}$. Subtracting β from $\mathcal{G}$-intensities of Z^B (resp. Z^S) in Lemma 2.5.1, we can read out intensities of $X^{B,B} - X^{S,B}$ (resp. $X^{S,S} - X^{B,S}$). Since property ii) at the beginning of this section implies that θ^B and θ^S are never positive at the same time. Therefore, when the intensity of $X^{B,B} - X^{S,B}$ is positive, the insider contributes buy orders $X^{B,B}$ with such intensity, otherwise the insider submits sell orders $X^{S,B}$ with the same intensity to cancel some noise buy orders from Z^B. Applying the same strategy to $X^{S,S} - X^{B,S}$ and utilizing Lemma 2.5.2, we read out $\mathscr{F}^I$-intensities for $X^{i,j}$, $i, j \in \{B, S\}$:

Corollary 2.5.3 *Suppose that $\mathscr{F}^I$-intensities of Y^B and Y^S match $\mathcal{G}$-intensities of Z^B and Z^S respectively, moreover $X_t^{B,B} = X_t^{B,S} \equiv 0$ when $Y_{t-} \ge \overline{m}_n$ and $X_t^{S,S} = X_t^{S,B} \equiv 0$ when $Y_{t-} \le \underline{m}_n$. Then $\mathscr{F}^I$-intensities of $X^{i,j}$, $i, j \in \{B, S\}$, have the following form on A_n when $Y_{t-} = y$:*

$$\theta^{B,B}(y,t) = \left(\frac{h_n(y+1,t)}{h_n(y,t)} - 1\right)_+ \beta, \qquad \theta^{B,S}(y,t) = \left(\frac{h_n(y-1,t)}{h_n(y,t)} - 1\right)_- \beta,$$

$$\theta^{S,S}(y,t)=\left(\frac{h_n(y-1,t)}{h_n(y,t)}-1\right)_+\beta, \qquad \theta^{S,B}(y,t)=\left(\frac{h_n(y+1,t)}{h_n(y,t)}-1\right)_-\beta.$$

In particular, $\theta^{i,j}$, $i,j\in\{B,S\}$, satisfies the following properties:

i) $\theta^{B,B}(y,\cdot)=\theta^{B,S}(y,\cdot)\equiv 0$, $\theta^{S,S}(y,\cdot)>0$, *and* $\theta^{S,B}(y,\cdot)>0$, *when* $y\geq \overline{m}_n$; $\theta^{S,S}(y,\cdot)=\theta^{S,B}(y,\cdot)\equiv 0$, $\theta^{B,B}(y,\cdot)>0$, *and* $\theta^{B,S}(y,\cdot)>0$, *when* $y\leq \underline{m}_n$;

ii) $\theta^{B,B}(\cdot,\cdot)=\theta^{S,S}(2m_n-\cdot,\cdot)$, $\theta^{B,S}(\cdot,\cdot)=\theta^{S,B}(2m_n-\cdot,\cdot)$;

iii) $\theta^{B,B}(\underline{m}_n,\cdot)=\theta^{S,S}(\overline{m}_n,\cdot)\equiv 0$.

As described in Corollary 2.5.3, when $A_n\in\overline{\mathscr{F}}_0$ is fixed, the state space is divided into two domains $\mathcal{S}:=\{y\in\mathbb{Z}:y\geq\overline{m}_n\}$ and $\mathcal{B}:=\{y\in\mathbb{Z}:y\leq\underline{m}_n\}$. As Y making excursions into these two domains, either X^S or X^B is active. In the following construction, we will focus on the domain $\mathcal{B}$ and construct inductively jumps of X^B until Y leaves $\mathcal{B}$. When Y excusers in $\mathcal{S}$, X^S can be constructed similarly.

When Y is in $\mathcal{B}$, one of the goals of X^B is to make sure that Y_1 ends up in the interval $[a_n,a_{n+1})$. In order to achieve this goal, X^B will add some jumps in addition to the jumps coming from Z^B. However this by itself will not be enough since Y also jumps downward due to Z^S. Thus, X^B also needs to cancel some of downwards jumps from Z^S. Therefore X^B consists of two components $X^{B,B}$ and $X^{B,S}$, where $X^{B,B}$ complements jumps of Z^B and $X^{B,S}$ cancels some jumps of Z^S. Let us denote by $(\tau_i)_{i\geq1}$ the sequence of jump times for Y. These stopping times will be constructed inductively as follows. Given $\tau_{i-1}<1$ and $Y_{\tau_{i-1}}\leq\underline{m}_n$, the next jump time τ_i happens at the minimum of the following three random times:

- the next jump of Z^B,
- the next jump of $X^{B,B}$,
- the next jump of Z^S which is not canceled by a jump of $X^{B,S}$.

Here $X^{B,B}$ and $X^{B,S}$ need to be constructed so that their intensities $\theta^{B,B}(Y_{t-},t)$ and $\theta^{B,S}(Y_{t-},t)$ match the forms in Corollary 2.5.3. This goal is achieved by employing two independent sequences of iid random variables $(\eta_i)_{i\geq1}$ and $(\zeta_i)_{i\geq1}$ with uniform distribution on $[0,1]$. They are also independent of $\overline{\mathcal{F}}$ and $(A_n)_{n=1,\cdots,N}$. These two sequences will be used to generate a random variable ν_i and another sequence of Bernoulli random variables $(\xi_{j,i})_{j\geq1}$ taking values in $\{0,1\}$. Let $(\sigma_i^+)_{i\geq1}$ and $(\sigma_i^-)_{i\geq1}$ be jump time of Z^B and Z^S, respectively. Then, after τ_{i-1}, the next jump of Z^B is at $\sigma^+_{Z^B_{\tau_{i-1}}+1}$, the next jump of $X^{B,B}$ is at ν_i, and the next jump of Z^S not canceled by jumps of $X^{B,S}$ is at $\tau_i^-=\min\{\sigma_j^->\tau_{i-1}:\xi_{j,i}=1\}$. Then the next jump of Y is at

$$\tau_i=\sigma^+_{Z^B_{\tau_{i-1}}+1}\wedge\nu_i\wedge\tau_i^-.$$

The construction of ν_i and $(\xi_{j,i})_{j\geq 1}$ using $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ is exactly the same as in [18, Section 4], only replacing h therein by h_n.

All aforementioned construction is performed in a filtered probability space $(\Omega, \mathscr{F}^I, (\mathscr{F}^I_t)_{t\in[0,1]}, \mathbb{P})$ such that there exist $(A_n)_{n=1,\cdots,N} \in \mathscr{F}^I_0$ with $\mathbb{P}(A_n) = h_n(0,0)$ and two independent sequences of iid $\mathscr{F}^I$-measurable random variables $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ with uniform distribution on $[0,1]$, moreover these two sequences are independent of both Z and $(A_n)_{n=1,\cdots,N}$. These requirements can be satisfied by extending $\overline{\mathscr{F}}_0$ (resp. $\overline{\mathscr{F}}$) to $\mathscr{F}^I_0$ (resp. $\mathscr{F}^I$). As for the filtration $(\mathscr{F}^I_t)_{t\in[0,1]}$, we require that it is right continuous and complete under $\mathbb{P}$, moreover Z, as the difference of two independent Poisson processes with intensity β, is adapted to $(\mathscr{F}^I_t)_{t\in[0,1]}$. Therefore Z is independent of $(A_n)_{n=1,\cdots,N}$, since Z has independent increments. Finally, we also assume that $(\mathscr{F}^I_t)_{t\in[0,1]}$ is rich enough so that $(\nu_i)_{i\geq 1}$ and $(\tau_i^-)_{i\geq 1}$ discussed above are $\mathscr{F}^I$-stopping times. An argument similar to [18, Lemma 4.3] yields:

Lemma 2.5.4 *Given point processes $(X^B, X^S; \mathscr{F}^I)$ constructed above, the $\mathscr{F}^I$-intensities of Y^B and Y^S at $t \in [0,1)$ are given by*

$$\sum_{n=1}^{N} \mathbb{I}_{\{A_n\}} \frac{h_n(Y_{t-}+1,t)}{h_n(Y_{t-},t)}\beta \quad \text{and} \quad \sum_{n=1}^{N} \mathbb{I}_{\{A_n\}} \frac{h_n(Y_{t-}-1,t)}{h_n(Y_{t-},t)}\beta,$$

respectively.

Now we are ready to verify that our construction is as desired.

Proposition 2.5.5 *The process Y as constructed above satisfies the following properties:*

i) $[Y_1 \in [a_n, a_{n+1})] = A_n$ *a.s. for* $n = 1, \cdots, N$*;*

ii) Y^B *and* Y^S *are independent Poisson processes with intensity* β *with respect to the natural filtration* $(\mathscr{F}^Y_t)_{t\in[0,1]}$ *of* Y*;*

iii) $(X^B, X^S; \mathscr{F}^I)$ *is admissible in the sense of Definition 2.2.2.*

Proof. To verify that Y satisfies the desired properties, let us introduce an auxiliary process $(\ell_t)_{t\in[0,1)}$:

$$\ell_t := \sum_{n=1}^{N} \mathbb{I}_{\{A_n\}} \frac{h_n(0,0)}{h_n(Y_t,t)}, \quad t \in [0,1).$$

When $n = 2, \cdots, N-1$, there is only almost surely finite number of positive (resp. negative) jumps of Y on A_n when $Y_\cdot \geq \overline{m}_n$ (resp. $Y_\cdot \leq \underline{m}_n$). Therefore Y_t is finite on these A_n when $t < 1$ is fixed. When $n = 1$ (resp. $n = N$), there is finite number of positive (resp. negative) jumps of Y on A_1 (resp. A_N) before t. Hence $Y_t < \infty$ on A_1 (resp. $Y_t > -\infty$ on A_N). This analysis implies

$h_n(Y_t, t) > 0$ on A_n for each $n = 1, \cdots, N$ and $t < 1$. Therefore $(\ell_t)_{t\in[0,1)}$ is well defined positive process with $\ell_0 = 1$.

To prove i), we first show that ℓ is a positive $\mathscr{F}^I$-local martingale on $[0, 1)$. To this end, Itô formula yields that

$$d\ell_t = \sum_{n=1}^N \mathbb{I}_{\{A_n\}}\ell_{t-} \left[\frac{h_n(Y_{t-}, t) - h_n(Y_{t-} + 1, t)}{h_n(Y_{t-} + 1, t)} dM_t^B + \frac{h_n(Y_{t-}, t) - h_n(Y_{t-} - 1, t)}{h_n(Y_{t-} - 1, t)} dM_t^S\right],$$

where $t \in [0, 1)$. Here

$$M^B = Y^B - \beta \int_0^{\cdot} \sum_{n=1}^N \mathbb{I}_{\{A_n\}} \frac{h_n(Y_{r-} + 1, r)}{h_n(Y_{r-}, r)} dr,$$
$$M^S = Y^S - \beta \int_0^{\cdot} \sum_{n=1}^N \mathbb{I}_{\{A_n\}} \frac{h_n(Y_{r-} - 1, r)}{h_n(Y_{r-}, r)} dr,$$

are all $\mathscr{F}^I$-local martingales. Define $\zeta_m^+ = \inf\{t \in [0, 1] : Y_t = m\}$ and $\zeta_m^- = \inf\{t \in [0, 1] : Y_t = -m\}$. Consider the sequence of stopping time $(\eta_m)_{m\geq1}$:

$$\eta_m := \left(\mathbb{I}_{\{\cup_{n=2}^{N-1} A_n\}} \zeta_m^+ \wedge \zeta_m^- + \mathbb{I}_{\{A_1\}}\zeta_m^+ + \mathbb{I}_{\{A_N\}}\zeta_m^-\right) \wedge (1 - 1/m).$$

It follows from the definition of h_n that each $h_n(Y_t, t)$ on A_n is bounded away from zero uniformly in $t \in [0, \eta_m]$. This implies that ℓ^{η_m} is bounded, hence ℓ^{η_m} is an $\mathscr{F}^I$-martingale. The construction of Y yields $\lim_{m\to\infty} \eta_m = 1$. Therefore, ℓ is a positive $\mathscr{F}^I$-local martingale, hence also a supermartingale, on $[0, 1)$.

Define $\ell_1 := \lim_{t\to1} \ell_t$, which exists and is finite due to Doob's supermartingale convergence theorem. This implies $h_n(Y_{1-}, 1) > 0$ on A_n. On the other hand, the construction of Y yields Y^S (resp. Y^B) does not jump at time 1 $\mathbb{P}$-a.s. when $Y_{1-} \leq \underline{m}_n$ (resp. $Y_{1-} \geq \overline{m}_n$). Therefore $h_n(Y_1, 1) > 0$ on A_n. However $h_n(\cdot, 1)$ by definition can only be either 0 or 1. Hence $Y_1 \in [a_n, a_{n+1})$ on A_n, for each $n = 1, \cdots, N$, and the statement i) is confirmed.

As for the statement ii), we will prove that Y^B is an $\mathscr{F}^Y$-adapted Poisson process. The similar argument can be applied to Y^S as well. In view of the $\mathscr{F}^I$-intensity of Y^B calculated in Lemma 2.5.4, one has that, for each $i \geq 1$,

$$Y^B_{\cdot\wedge\tau_i\wedge1} - \beta \left(\int_0^{\cdot\wedge\tau_i\wedge1} \sum_{n=1}^N \mathbb{I}_{\{A_n\}} \frac{h_n(Y_{u-} + 1, u)}{h_n(Y_{u-}, u)} du\right)$$

is an $\mathscr{F}^I$-martingale, where τ_i is the i^{th} jump time of Y. We will show in the next paragraph that, when stopped at $\tau_i \wedge 1$, Y^B is Poisson process in $\mathscr{F}^Y$ by showing that $(Y^B_{\tau_i\wedge t} - \beta(\tau_i \wedge t))_{t\in[0,1]}$ is an $\mathscr{F}^Y$-martingale. (Here note that τ_i is an $\mathscr{F}^Y$-stopping time.) This in turn will imply that Y^B is a Poisson process with intensity β on $[0, \tau \wedge 1)$ where $\tau = \lim_{i\to\infty} \tau_i$ is the explosion time. Since Poisson process does not explode, this will further imply $Y^B_{\tau\wedge1} < \infty$ and, therefore, $\tau \geq 1$, $\mathbb{P}$-a.s..

We proceed by projecting the above martingale into $\mathscr{F}^Y$ to see that

$$Y^B - \beta \int_0^{\cdot} \sum_{n=1}^{N} \mathbb{P}(A_n|\mathscr{F}_r^Y) \frac{h_n(Y_{r-}+1,r)}{h_n(Y_{r-},r)} dr$$

is an $\mathscr{F}^Y$-martingale when stopped at $\tau_i \wedge 1$. Therefore, it remains to show that, for almost all $t \in [0,1)$, on $[t \le \tau_i]$,

$$\sum_{n=1}^{N} \mathbb{P}(A_n|\mathscr{F}_t^Y) \frac{h_n(Y_{t-}+1,t)}{h_n(Y_{t-},t)} = 1, \quad \mathbb{P}\text{-a.s.}. \tag{2.5.3}$$

To this end, we will show, on $[t \le \tau_i]$,

$$\mathbb{P}(A_n|\mathscr{F}_t^Y) = h_n(Y_t, t), \quad \text{for } t \in [0,1). \tag{2.5.4}$$

Then (2.5.3) follows since $Y_t \neq Y_{t-}$ only for countably many times.

We have seen that $(\ell_{u\wedge\tau_i})_{u\in[0,t]}$ is a strictly positive $\mathscr{F}^I$-martingale for each i. Define a probability measure $\mathbb{Q}^i \sim \mathbb{P}$ on $\mathscr{F}_t^I$ via $d\mathbb{Q}^i/d\mathbb{P}|_{\mathscr{F}_t^I} = \ell_{\tau_i\wedge t}$. It follows from Girsanov's theorem that Y^B is a Poisson process when stopped at $\tau_i \wedge t$ and with intensity β under $\mathbb{Q}^i$. Therefore, they are independent from A_n under $\mathbb{Q}^i$. Then, for $t < 1$, we obtain from the Bayes's formula that

$$\begin{aligned}
\mathbb{I}_{\{r\le\tau_i\wedge t\}}\mathbb{P}(A_n|\mathscr{F}_r^Y) &= \mathbb{I}_{\{r\le\tau_i\wedge t\}} \frac{\mathbb{E}^{\mathbb{Q}^i}[\mathbb{I}_{\{A_n\}}\ell_r^{-1}|\mathscr{F}_r^Y]}{\mathbb{E}^{\mathbb{Q}^i}[\ell_r^{-1}|\mathscr{F}_r^Y]} \\
&= \mathbb{I}_{\{r\le\tau_i\wedge t\}} \frac{\mathbb{E}^{\mathbb{Q}^i}[\mathbb{I}_{\{A_n\}}\frac{h_n(Y_r,r)}{h_n(0,0)}|\mathscr{F}_r^Y]}{\mathbb{E}^{\mathbb{Q}^i}[\sum_{n=1}^{N}\mathbb{I}_{\{A_n\}}\frac{h_n(Y_r,r)}{h_n(0,0)}|\mathscr{F}_r^Y]} \\
&= \mathbb{I}_{\{r\le\tau_i\wedge t\}} h_n(Y_r, r),
\end{aligned} \tag{2.5.5}$$

where the third identity follows from the aforementioned independence of Y and A_n under $\mathbb{Q}^i$ along with the fact that $\mathbb{Q}^i$ does not change the probability of $\mathscr{F}_0^I$ measurable events so that $\mathbb{Q}^i(A_n) = \mathbb{P}(A_n) = h_n(0,0)$. As result, (2.5.4) follows from (2.5.5) after sending $i \to \infty$.

Since Y^B and Y^S are $\mathscr{F}^Y$-Poisson processes and they do not jump simultaneously by their construction, they are then independent. To show the strategy $(X^B, X^S; \mathscr{F}^I)$ constructed is admissible, it remains to show both $\mathbb{E}[X_1^B\mathbb{I}_{\{A_n\}}]$ and $\mathbb{E}[X_1^S\mathbb{I}_{\{A_n\}}]$ are finite for each $n = 1, \cdots, N$. To this end, for each n, $\mathbb{E}[X_1^B\mathbb{I}_{\{A_n\}}] = \mathbb{E}[X_1^{B,B}\mathbb{I}_{\{A_n\}}] + \mathbb{E}[X_1^{B,S}\mathbb{I}_{\{A_n\}}]$, where $\mathbb{E}[X_1^{B,S}\mathbb{I}_{\{A_n\}}] \le \mathbb{E}[Z^S] < \infty$ and $\mathbb{E}[X_1^{B,B}\mathbb{I}_{\{A_n\}}] \le \mathbb{E}[Y_1^B\mathbb{I}_{\{A_n\}}] + \mathbb{E}[X_1^{S,B}\mathbb{I}_{\{A_n\}}] \le \mathbb{E}[Z_1^B \,|\, Z \in [a_n, a_{n+1})] + \mathbb{E}[Z_1^S] < \infty$. Similar argument also implies $\mathbb{E}[X_1^S\mathbb{I}_{\{A_n\}}] < \infty$. Finally, since $N < \infty$, p is bounded, Definition 2.2.2 iv) is verified using $\mathbb{E}[X_1^B\mathbb{I}_{\{A_n\}}], \mathbb{E}[X_1^S\mathbb{I}_{\{A_n\}}] < \infty$ for each $n \in \{1, \ldots, N\}$. $\square$

2.6 Convergence

Collecting results from previous sections, we will prove Theorems 2.2.12 and 2.2.13 in this section. Let us first construct a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$, each of which will be the fundamental value in the Glosten-Milgrom model with order size δ.

Adding to the sequence of canonical spaces $(\Omega^\delta, \mathscr{F}^{Z,\delta}, (\mathscr{F}_t^{Z,\delta})_{t\in[0,1]}, \mathbb{P}^\delta)$, defined at the beginning of section 2.2.2, we introduce $(\Omega^0, \mathscr{F}^0, (\mathscr{F}_t^0)_{t\in[0,1]}, \mathbb{P}^0)$, where $\Omega^0 = \mathbb{D}([0,1], \mathbb{R})$ is the space of $\mathbb{R}$-valued càdlàg functions on $[0,1]$ with coordinate process Z^0, and $\mathbb{P}^0$ is the Wiener measure. Denote by $\mathbb{P}^{0,y}$ the Wiener measure under which $Z_0^0 = y$ a.s.. Let us now define a $\mathbb{R}\cup\{-\infty,\infty\}$-valued sequence $(a_n^0)_{n=1,\cdots N+1}$ via

$$a_1^0 = -\infty, \quad a_n^0 = \Phi^{-1}\left(p_1 + \cdots + p_{n-1}\right), \quad n = 2, \cdots, N+1,$$

where $\Phi(\cdot) = \int_{-\infty}^{\cdot} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$. Using this sequence, one can define a pricing rule following the same recipe in (2.2.5):

$$p^0(y,t) := \sum_{n=1}^{N} v_n h_n^0(y,t), \quad y \in \mathbb{R}, \quad t \in [0,1], \quad n \in \{1, \cdots, N\}, \tag{2.6.1}$$

where $h_n^0(y,t) := \mathbb{P}^{0,y}\left(Z_{1-t}^0 \in [a_n^0, a_{n+1}^0)\right) = \Phi(a_{n+1}^0 - y) - \Phi(a_n^0 - y)$.

As we will see later, this is exactly the pricing rule in the Kyle-Back equilibrium. Moreover, the sequence $(a_n^\delta)_{n=1,\cdots,N+1}$, associated to $(\tilde{v}^\delta)_{\delta>0}$ constructed below, converges to $(a_n^0)_{n=1,\cdots,N+1}$ as $\delta \downarrow 0$, helping to verify Definition 2.2.11 i).

Lemma 2.6.1 *For any $\tilde{v}$ with distribution* (2.1.1) *where N may not be finite, there exists a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$, each of which takes value in $\{v_1, \cdots, v_N\}$, such that*

i) Assumption 2.4.1 is satisfied when $\tilde{v}$ therein is replaced by each $\tilde{v}^\delta$ [10]*;*

ii) $Law(\tilde{v}^\delta) \implies Law(\tilde{v})$, as $\delta \downarrow 0$. Here $\implies$ represents the weak convergence of probability measures.

Proof. For each $\delta > 0$, $\tilde{v}^\delta$ will be constructed by adjusting p_n in (2.1.1) to some p_n^δ, $n = 1, \cdots, N$. Starting from $[\tilde{v} = v_1]$, choose $a_1^\delta = -\infty$, $a_2^\delta = \inf\{y \in \delta\mathbb{Z} : \mathbb{P}^\delta(Z_1^\delta \le y) \ge p_1\}$, and set $\mathbb{P}^\delta(\tilde{v}^\delta = v_1) = \mathbb{P}^\delta(Z_1^\delta \in [a_1^\delta, a_2^\delta))$. Moving on to $[\tilde{v}^\delta = v_2]$, choose $a_3^\delta = \inf\{y \in \delta\mathbb{Z} : \mathbb{P}^\delta(Z_1^\delta \le y) \ge p_1 + p_2 \text{ and } (a_2^\delta + y - \delta)/2 \notin \delta\mathbb{Z}\}$ and set $\mathbb{P}^\delta(\tilde{v}^\delta = v_2) = \mathbb{P}^\delta(Z_1^\delta \in [a_2^\delta, a_3^\delta))$. Following this step, we can define a_n^δ inductively. When $N < \infty$, we set $a_{N+1}^\delta = \infty$. This construction gives a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$ taking values in $\{v_1, \cdots, v_N\}$ such that $\mathbb{P}^\delta(\tilde{v}^\delta = v_n) = p_n^\delta := \mathbb{P}^\delta(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta))$ with $\sum_{n=1}^N p_n^\delta = 1$, moreover each sequence $(a_n^\delta)_{n=1,\cdots,N+1}$ satisfies Assumption 2.4.1.

It remains to show $Law(\tilde{v}^\delta) \implies Law(\tilde{v})$ as $\delta \downarrow 0$. To this end, note that a_n^δ is either the $(\sum_{i=1}^{n-1} p_i)^{th}$ quantile of the distribution of Z_1^δ or δ above this quantile. When β^δ is chosen as $1/(2\delta^2)$, it follows from [24, Chapter 6, Theorem 5.4] that $\mathbb{P}^\delta \implies \mathbb{P}^0$, in particular, $Law(Z_1^\delta) \implies Law(Z_1^0)$. Therefore,

$$\lim_{\delta\downarrow 0} a_n^\delta = a_n^0, \quad n = 1, \cdots, N+1. \tag{2.6.2}$$

[10] When the order size is δ, Assumption 2.4.1 iii) reads $(a_n^\delta + a_{n+1}^\delta - \delta)/2 \notin \delta\mathbb{Z}$.

For any $\epsilon > 0$ and $n \in \{1, \cdots, N\}$, the previous convergence yields the existence of a sufficiently small $\delta_{\epsilon,n}$, such that $[a_n^0 + \epsilon, a_{n+1}^0 - \epsilon) \subseteq [a_n^\delta, a_{n+1}^\delta) \subseteq [a_n^0 - \epsilon, a_{n+1}^0 + \epsilon)$ for any $\delta \leq \delta_{\epsilon,n}$. Hence

$$\mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) \leq \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^0 - \epsilon, a_{n+1}^0 + \epsilon)\right) \to \mathbb{P}^0\left(Z_1^0 \in [a_n^0 - \epsilon, a_{n+1}^0 - \epsilon)\right),$$
$$\mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) \geq \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^0 + \epsilon, a_{n+1}^0 - \epsilon)\right) \to \mathbb{P}^0\left(Z_1^0 \in [a_n^0 + \epsilon, a_{n+1}^0 - \epsilon)\right),$$

as $\delta \downarrow 0$, where both convergence follow from $Law(Z_1^\delta) \Longrightarrow Law(Z_1^0)$ and the fact that the distribution of Z_1^0 is continuous. Since ϵ is arbitrarily chosen, utilizing the continuity of the distribution for Z_1^0 again, we obtain from the previous two inequalities

$$\lim_{\delta\downarrow 0} \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) = \mathbb{P}^0\left(Z_1^0 \in [a_n^0, a_{n+1}^0)\right).$$

Hence $\lim_{\delta\downarrow 0} p_n^\delta = p_n^0$ for each $n \in \{1, \cdots N\}$ and $Law(\tilde{v}^\delta) \Rightarrow Law(\tilde{v})$. □

After $(\tilde{v}^\delta)_{\delta>0}$ is constructed, it follows from Sections 2.4 and 2.5 that a sequence of strategies $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})_{\delta>0}$ exists, each of which satisfies conditions in Proposition 2.4.5. Hence p^δ in (2.2.5) is rational for each $\delta > 0$. It then remain to verify Definition 2.2.11 iii) to establish an asymptotic Glosten-Milgrom equilibrium.

Before doing this, we prove Theorem 2.2.13 first. Let us recall the Kyle-Back equilibrium. Following arguments in [32] and [5], the equilibrium pricing rule is given by (2.6.1) and the equilibrium demand satisfies the SDE

$$Y^0 = Z^0 + \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}=v_n\}} \int_0^\cdot \frac{\partial_y h_n^0(Y_r^0, r)}{h_n^0(Y_r^0, r)}\, dr,$$

where Z^0 is a $\mathbb{P}^0$-Brownian motion modeling the demand from noise traders. Hence the insider's strategy in the Kyle-Back equilibrium is given by

$$X^0 = \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}=v_n\}} \int_0^\cdot \frac{\partial_y h_n^0(Y_r^0, r)}{h_n^0(Y_r^0, r)}\, dr.$$

Proof of Theorem 2.2.13. As we have seen in Lemma 2.6.1, Assumption 2.4.1 is satisfied by each $\tilde{v}^\delta$. It then follows from Proposition 2.5.5 i) and ii) that the distribution of Y^δ on $[\tilde{v}^\delta = v_n]$ is the same as the distribution of Z^δ conditioned on $Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)$. Denote $Y^{0,n} = Y^0 \mathbb{I}_{\{\tilde{v}=v_n\}}$ as the cumulative demand in Kyle Back equilibrium when the fundamental value is v_n. The same argument as in [18, Lemma 5.4] yields

$$Law(Z^\delta \mid Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)) \Longrightarrow Law(Y^{0,n}), \quad \text{as } \delta \downarrow 0,$$

for each $n \in \{1, \cdots, N\}$. It then follows

$$Law(Y^\delta; \mathscr{F}^{I,\delta}) \Longrightarrow Law(Y^0; \mathscr{F}^{I,0}), \quad \text{as } \delta \downarrow 0, \tag{2.6.3}$$

where the filtration $\mathscr{F}^{I,0}$ is $\mathscr{F}^0$ initially enlarged by $\tilde{v}$. Recall from (2.5.1) that $Y^\delta = Z^\delta + X^{B,\delta} - X^{S,\delta}$, moreover $Y^0 = Z^0 + X^0$. Combining (2.6.3) with $Law(Z^\delta) \Longrightarrow Law(Z^0)$, we conclude from [30, Proposition VI.1.23] that $Law(X^{B,\delta} - X^{S,\delta}) \Longrightarrow Law(X^0)$ as $\delta \downarrow 0$. $\square$

In the rest of the section, Definition 2.2.11 iii) is verified for strategies $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})_{\delta>0}$, which concludes the proof of Theorem 2.2.12. We have seen in Proposition 2.4.2 that the expected profit of the strategy $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})$, constructed in section 2.5, satisfies

$$\mathcal{J}^\delta(v_n, 0, 0; X^{B,\delta}, X^{S,\delta}) = U^\delta(v_n, 0, 0) - L^\delta(v_n, 0, 0), \quad n \in \{1, \cdots, N\},$$

where

$$\begin{aligned} L^\delta(v_n, 0, 0) = \delta\beta^\delta\, \mathbb{E}^{\delta,0} & \left[\int_0^1 (v_n - p^\delta(\underline{m}_n^\delta, r))\, \mathbb{I}_{\{Y^\delta_{r-} = \overline{m}_n^\delta\}} dr \,\middle|\, \tilde{v}^\delta = v_n \right] \\ - \delta\beta^\delta\, \mathbb{E}^{\delta,0} & \left[\int_0^1 (v_n - p^\delta(\overline{m}_n^\delta, r))\, \mathbb{I}_{\{Y^\delta_{r-} = \underline{m}_n^\delta\}} dr \,\middle|\, \tilde{v}^\delta = v_n \right]. \end{aligned} \tag{2.6.4}$$

This expression for L^δ follows from changing the order size in (2.4.7) from 1 to δ and utilizing $\theta^{B,S,\delta}(\overline{m}_n^\delta, \cdot) = \theta^{S,S,\delta}(\overline{m}_n^\delta, \cdot) = \theta^{S,B,\delta}(\underline{m}_n^\delta, \cdot) = \theta^{B,B,\delta}(\underline{m}_n^\delta, \cdot) = 0$ from Corollary 2.5.3 i) and iii), $\theta^{B,T,\delta} = \theta^{S,T,\delta} \equiv 0$ from Remark 2.4.6, and the expectations are taken under $\mathbb{P}^{\delta,0}$. Here $\underline{m}_n^\delta := \delta\lfloor (a_n + a_{n+1} - \delta)/2\delta \rfloor$ the largest integer multiple of δ smaller than m_n^δ and by $\overline{m}_n^\delta := \delta\lceil (a_n + a_{n+1} - \delta)/2\delta \rceil$ the smallest integer multiple of δ larger than m_n^δ. To prove Theorem 2.2.13, let us first show

$$\lim_{\delta \downarrow 0} L^\delta(v_n, 0, 0) = 0, \quad n \in \{1, \cdots, N\}. \tag{2.6.5}$$

In the rest development, we fix v_n and denote $L^\delta = L^\delta(v_n, 0, 0)$.

Before presenting technical proofs for (2.6.5), let us first introduce a heuristic argument. First, since $\beta^\delta = 1/(2\delta^2)$, (2.6.4) can be rewritten as

$$L^\delta = \mathbb{E}^{\delta,0}\left[\overline{I}_1^{\delta,n} \,\middle|\, \tilde{v}^\delta = v_n \right] - \mathbb{E}^{\delta,0}\left[\underline{I}_1^{\delta,n} \,\middle|\, \tilde{v}^\delta = v_n \right], \tag{2.6.6}$$

where

$$\overline{I}_\cdot^{\delta,n} = \int_0^\cdot (v_n - p^\delta(Y^\delta_{r-} - \delta, r))\, d\mathcal{L}_r^{\delta,\overline{m}_n^\delta}, \quad \underline{I}_\cdot^{\delta,n} = \int_0^\cdot (v_n - p^\delta(Y^\delta_{r-} + \delta, r))\, d\mathcal{L}_r^{\delta,\underline{m}_n^\delta},$$

and $\mathcal{L}_\cdot^{\delta,y} = \frac{1}{2\delta}\int_0^\cdot \mathbb{I}_{\{Y^\delta_{r-} = y\}} dr$ is the *scaled occupation time* of Y^δ at level y. Here Y^δ is, in its natural filtration, the difference of two independent Poisson $Y^{B,\delta}$ and $Y^{S,\delta}$ with jump size δ and intensity β^δ, cf. Proposition 2.5.5 ii). For the integrands in $\overline{I}^{\delta,n}$ and $\underline{I}^{\delta,n}$, we expect that $v_n - p^\delta(Y_\cdot^\delta \pm \delta, \cdot) \xrightarrow{\mathcal{L}} v_n - p^0(Y_\cdot^0, \cdot)$, where Y^0 is a $\mathbb{P}^0$-Brownian motion. As for the integrators, we will show both $\mathcal{L}_\cdot^{\delta,\overline{m}_n^\delta}$ and $\mathcal{L}_\cdot^{\delta,\underline{m}_n^\delta}$ converge weakly to $\mathcal{L}_\cdot^{m_n}$, which is the Brownian local time at level $m_n := (a_n^0 + a_{n+1}^0)/2$. Then the weak convergence of both integrands and integrators yield

$$\overline{I}_\cdot^{\delta,n} \text{ and } \underline{I}_\cdot^{\delta,n} \xrightarrow{\mathcal{L}} I_\cdot^{0,n} := \int_0^\cdot (v_n - p^0(Y_r^0, r))\, d\mathcal{L}_r^{m_n}, \quad \text{as } \delta \downarrow 0.$$

Finally passing the previous convergence to conditional expectation, the two terms on the right hand side of (2.6.6) cancel each other in the limit.

Proposition 2.6.2 *On the family of filtration $(\mathscr{F}^{Y,\delta}_t)_{t\in[0,1],\delta\geq 0}$, generated by $(Y^\delta)_{\delta\geq 0}$,*

$$p^\delta(Y^\delta_\cdot \pm \delta, \cdot) \xrightarrow{\mathcal{L}} p^0(Y^0_\cdot, \cdot) \quad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

Proof. To simplify presentation, we will prove

$$p^\delta(Y^\delta_\cdot, \cdot) \xrightarrow{\mathcal{L}} p^0(Y^0_\cdot, \cdot) \quad \text{as } \delta \downarrow 0. \tag{2.6.7}$$

The assertions with $\pm\delta$ can be proved by replacing Y^δ by $Y^\delta \pm \delta$. First, applying Itô's formula and utilizing (2.4.1) yield

$$\begin{aligned} p^\delta(Y^\delta_\cdot, \cdot) = p^\delta(0,0) &+ \int_0^\cdot \frac{1}{\delta}\left(p^\delta(Y^\delta_{r-}+\delta, r) - p^\delta(Y^\delta_{r-}, r)\right) d\overline{Y}^{B,\delta}_r \\ &+ \int_0^\cdot \frac{1}{\delta}\left(p^\delta(Y^\delta_{r-}-\delta, r) - p^\delta(Y^\delta_{r-}, r)\right) d\overline{Y}^{S,\delta}_r, \end{aligned} \tag{2.6.8}$$

where $\overline{Y}^{B,\delta}_\cdot = Y^{B,\delta}_\cdot - \delta\beta^\delta\cdot$ and $\overline{Y}^{S,\delta}_\cdot = Y^{S,\delta}_\cdot - \delta\beta^\delta\cdot$ are compensated jump processes. For $p^\delta(0,0)$ on the right hand side, the same argument in Lemma 2.6.1 yields $\lim_{\delta\downarrow 0} p^\delta(0,0) = p^0(0,0)$. As for the other two stochastic integrals, we will show that they converge weakly to

$$\frac{1}{\sqrt{2}}\int_0^\cdot \partial_y p^0(Y^0_r, r) dW^B_r \quad \text{and} \quad -\frac{1}{\sqrt{2}}\int_0^\cdot \partial_y p^0(Y^0_r, r) dW^S_r, \quad \text{respectively,}$$

where W^B and W^S are two independent Brownian motion. These estimates then imply the right hand side of (2.6.8) converges weakly to

$$p^0(0,0) + \int_0^\cdot \partial_y p^0(Y^0_r, r)\, dW_r,$$

where $W = W^B/\sqrt{2} - W^S/\sqrt{2}$ is another Brownian motion. Since p^0 satisfies $\partial_t p^0 + \frac{1}{2}\partial^2_{yy} p^0 = 0$, the previous process has the same law as $p^0(Y^0_\cdot, \cdot)$. Therefore (2.6.7) is confirmed.

To prove the aforementioned convergence of stochastic integrals, let us first derive the convergence of $(p^\delta(\cdot+\delta, \cdot) - p^\delta(\cdot,\cdot))/\delta$ on $\mathbb{R}\times[0,1)$. To this end, it follows from (2.2.5) that

$$\begin{aligned} &\frac{1}{\delta}(p^\delta(y+\delta, t) - p^\delta(y,t)) \\ &= \frac{1}{\delta}\sum_{n=1}^N v_n \left[\mathbb{P}^{\delta, y+\delta}(Z^\delta_{1-t} \in [a^\delta_n, a^\delta_{n+1})) - \mathbb{P}^{\delta,y}(Z^\delta_{1-t} \in [a^\delta_n, a^\delta_{n+1}))\right] \\ &= \frac{1}{\delta}\sum_{n=1}^N v_n \left[\mathbb{P}^{\delta,y}(Z^\delta_{1-t} = a^\delta_n - \delta) - \mathbb{P}^{\delta,y}(Z^\delta_{1-t} = a^\delta_{n+1} - \delta)\right] \\ &= \frac{1}{\delta}\sum_{n=1}^N v_n \left[\mathbb{P}^{1,0}\left(Z^1_{1-t} = \frac{a^\delta_n - \delta - y}{\delta}\right) - \mathbb{P}^{1,0}\left(Z^1_{1-t} = \frac{a^\delta_{n+1} - \delta - y}{\delta}\right)\right] \end{aligned}$$

$$= \sum_{n=1}^{N} v_n \left[\frac{1}{\delta} e^{-\frac{1-t}{\delta^2}} I_{\left|\frac{a_n^\delta - \delta - y}{\delta}\right|}\left(\frac{1-t}{\delta^2}\right) - \frac{1}{\delta} e^{-\frac{1-t}{\delta^2}} I_{\left|\frac{a_{n+1}^\delta - \delta - y}{\delta}\right|}\left(\frac{1-t}{\delta^2}\right)\right]$$
$$\to \sum_{n=1}^{N} v_n \left[\frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(a_n^0 - y)^2}{2(1-t)}\right) - \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(a_{n+1}^0 - y)^2}{2(1-t)}\right)\right]$$
$$= \partial_y p^0(y,t), \quad \text{as } \delta \downarrow 0.$$

Here Z^1_{1-t} is the difference of two independent Poisson random variables with common parameter $(1-t)\beta^\delta = (1-t)(2\delta^2)^{-1}$ under $\mathbb{P}^{1,0}$. Hence the fourth identity above follows from the probability distribution function of the *Skellam distribution*: $\mathbb{P}^{1,0}(Z^1_{1-t} = k) = e^{-2\mu} I_{|k|}(2\mu)$, where $I_{|k|}(\cdot)$ is the *modified Bessel function of the second kind* and $\mu = (1-t)(2\delta^2)^{-1}$, cf. [40]. The convergence above is locally uniformly in $\mathbb{R} \times [0,1)$ according to [3, Theorem 2]. The last identity above follows from taking y derivative to $p^0(y,t) = \sum_{n=1}^N \left(\Phi\left(\frac{a^0_{n+1}-y}{\sqrt{1-t}}\right) - \Phi\left(\frac{a^0_n - y}{\sqrt{1-t}}\right)\right)$, cf. (2.6.1). Combining the previous locally uniform convergence of $(p^\delta(\cdot+\delta,\cdot) - p^\delta(\cdot,\cdot))/\delta$ with the weak convergence $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ in their natural filtration, we have from [10, Chapter 1, Theorem 5.5]:

$$\frac{1}{\delta}\left(p^\delta(Y^\delta_\cdot + \delta, \cdot) - p^\delta(Y^\delta_\cdot, \cdot)\right) \xrightarrow{\mathcal{L}} \partial_y p^0(Y^0_\cdot, \cdot) \quad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

As for the integrators in (2.6.8), $\overline{Y}^{B,\delta} \xrightarrow{\mathcal{L}} W^B/\sqrt{2}$ and $\overline{Y}^{S,\delta} \xrightarrow{\mathcal{L}} W^S/\sqrt{2}$. Moreover, both $(\overline{Y}^{B,\delta})_{\delta>0}$ and $(\overline{Y}^{S,\delta})_{\delta>0}$ are *predictable uniform tight* (P-UT), since $\langle \overline{Y}^{B,\delta}\rangle_t = \langle \overline{Y}^{S,\delta}\rangle_t = t/2$, for any $\delta > 0$, cf. [30, Chapter VI, Theorem 6.13 (iii)]. Then combining weak convergence of both integrands and integrators, we obtain from [30, Chapter VI, Theorem 6.22] that

$$\int_0^\cdot \frac{1}{\delta}(p^\delta(Y^\delta_{r-} + \delta, r) - p^\delta(Y^\delta_{r-}, r))\, d\overline{Y}^{B,\delta}_r \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{2}} \int_0^\cdot \partial_y p^0(Y^0_r, r)\, dW^B_r \text{ on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

A similar weak convergence holds for the other stochastic integral in (2.6.8) as well. Therefore the claimed weak convergence of stochastic integrals on the right hand side of (2.6.8) is confirmed. □

Having studied the weak convergence of integrands in $\overline{I}^{\delta,n}$ and $\underline{I}^{\delta,n}$, let us switch our attention to the integrators $\mathcal{L}^{\delta,\overline{m}^\delta_n}$ and $\mathcal{L}^{\delta,\underline{m}^\delta_n}$.

Proposition 2.6.3 *On the family of filtration* $(\mathscr{F}^{Y,\delta}_t)_{t\in[0,1],\delta\geq 0}$, *for any* $n \in \{1,\cdots,N\}$,

$$\mathcal{L}^{\delta,\overline{m}^\delta_n} \xrightarrow{\mathcal{L}} \mathcal{L}^{m_n} \quad \textit{and} \quad \mathcal{L}^{\delta,\underline{m}^\delta_n} \xrightarrow{\mathcal{L}} \mathcal{L}^{m_n} \quad \textit{on } \mathbb{D}[0,1] \textit{ as } \delta \downarrow 0.$$

Proof. For simplicity of presentation, we will prove

$$\mathcal{L}^{\delta,0} \xrightarrow{\mathcal{L}} \mathcal{L}^0 \quad \text{as } \delta \downarrow 0. \tag{2.6.9}$$

Since $\lim_{\delta\downarrow 0} \overline{m}^\delta_n = \lim_{\delta\downarrow 0} \underline{m}^\delta_n = m_n$ follows from (2.6.2), the statement of the proposition follows from replacing Y^δ by $Y^\delta - \overline{m}^\delta_n$ (or by $Y^\delta - \underline{m}^\delta_n$) and Y^0 by $Y^0 - m_n$ in the rest of the proof. To prove (2.6.9), applying Itô's formula to $|Y^\delta_\cdot|$ yields

$$
\begin{aligned}
|Y^\delta_\cdot| &= \sum_{r\le\cdot}\left(|Y^\delta_r| - |Y^\delta_{r-}|\right) \\
&= \int_0^\cdot \left(|Y^\delta_{r-}+\delta| - |Y^\delta_{r-}|\right) d(Y^{B,\delta}_r/\delta - \beta^\delta r) \\
&\quad + \int_0^\cdot \left(|Y^\delta_{r-}-\delta| - |Y^\delta_{r-}|\right) d(Y^{S,\delta}_r/\delta - \beta^\delta r) \\
&\quad + \int_0^\cdot \left(|Y^\delta_{r-}+\delta| + |Y^\delta_{r-}-\delta| - 2|Y^\delta_{r-}|\right)\beta^\delta dr \\
&= \int_0^\cdot \left(|Y^\delta_{r-}+\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{B,\delta}_r/\delta + \int_0^\cdot \left(|Y^\delta_{r-}-\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{S,\delta}_r/\delta \\
&\quad + \int_0^\cdot \frac{1}{\delta}\mathbb{I}_{\{Y^\delta_{r-}=0\}} dr,
\end{aligned}
\tag{2.6.10}
$$

where the third identity follows from $|y+\delta|+|y-\delta|-2|y| = 2\delta\,\mathbb{I}_{\{y=0\}}$ for any $y\in\mathbb{R}$. On the other hand, Tanaka formula for Brownian motion is

$$
|Y^0_\cdot| = \int_0^\cdot sgn(Y^0_r)\,dY^0_r + 2\mathcal{L}^0_\cdot, \tag{2.6.11}
$$

where $sgn(x)=1$ when $x>0$ or -1 when $x\le 0$.

The convergence (2.6.9) is then confirmed by comparing both sides of (2.6.10) and (2.6.11). To this end, since $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ and the absolute value is a continuous function, then $|Y^\delta| \xrightarrow{\mathcal{L}} |Y^0|$ follows from [10, Chapter 1, Theorem 5.1]. Then (2.6.9) is confirmed as soon as we prove the martingale term on the right hand side of (2.6.10) converges weakly to the martingale in (2.6.11), which we prove in the next result. □

Lemma 2.6.4 *Let* $M^\delta := \int_0^\cdot \left(|Y^\delta_{r-}+\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{B,\delta}_r/\delta + \int_0^\cdot \left(|Y^\delta_{r-}-\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{S,\delta}_r/\delta$ *and* $M^0 := \int_0^\cdot sgn(Y^0_r)\,dY^0_r$. *Then* $M^\delta \xrightarrow{\mathcal{L}} M^0$ *on* $\mathbb{D}[0,1]$ *as* $\delta\downarrow 0$.

Proof. Define $f^\delta(y) := \frac{1}{\delta}(|y+\delta|-|y|)$ for $y\in\mathbb{R}$ and observe

$$
f^\delta(y) = \begin{cases} 1 & y\ge 0 \\ 2y/\delta+1 & -\delta<y<0 \\ -1 & y\le -\delta \end{cases}.
$$

It is clear that f^δ converges to $sgn(\cdot)$ locally uniformly on $\mathbb{R}\setminus\{0\}$. On the other hand, $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ and the law of Y^0 is continuous. It then follows from [10, Chapter 1, Theorem 5.5] that $f^\delta(Y^\delta) \xrightarrow{\mathcal{L}} sgn(Y^0)$. As for the integrators $(\overline{Y}^{B,\delta})_{\delta>0}$, as we have seen in the proof of Proposition 2.6.2, they converge weakly to $W^B/\sqrt{2}$ and are P-UT. Then [30, Chapter VI, Theorem 6.22] implies

$$
\int_0^\cdot \left(|Y^\delta_{r-}+\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{B,\delta}_r/\delta \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{2}}\int_0^\cdot sgn(Y^0_r)\,dW^B_r.
$$

Similar argument yields

$$
\int_0^\cdot \left(|Y^\delta_{r-}-\delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{S,\delta}_r/\delta \xrightarrow{\mathcal{L}} -\frac{1}{\sqrt{2}}\int_0^\cdot sgn(Y^0_r)\,dW^S_r.
$$

Here W^B and W^S are independent Brownian motion. Defining $W = W^B/\sqrt{2} - W^S/\sqrt{2}$, we obtain from the previous two convergence that

$$M^\delta \xrightarrow{\mathcal{L}} \int_0^\cdot sgn(Y_r^0)\, dW_r \quad \text{which has the same law as} \quad M^0. \qquad \square$$

Propositions 2.6.2 and 2.6.3 combined yields the weak convergence of $(\overline{I}^{\delta,n})_{\delta>0}$ and $(\underline{I}^{\delta,n})_{\delta>0}$. Moreover the sequence of local time in Proposition 2.6.3 also converge in expectation.

Corollary 2.6.5 *On the family of filtration $(\mathscr{F}_t^{Y,\delta})_{t\in[0,1],\delta\geq 0}$, for any $n \in \{1, \cdots, N\}$,*

$$\overline{I}^{\delta,n} \text{ and } \underline{I}^{\delta,n} \xrightarrow{\mathcal{L}} I^{0,n} \qquad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

Proof. The statement follows from combining Propositions 2.6.2 and 2.6.3, and appealing to [30, Chapter VI, Theorem 6.22]. In order to apply the previous result, we need to show that both $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta>0}$ and $(\mathcal{L}^{\delta,\underline{m}_n^\delta})_{\delta>0}$ are P-UT. This property will be verified for $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta>0}$. The same argument works for $(\mathcal{L}^{\delta,\underline{m}_n^\delta})_{\delta>0}$ as well. To this end, since $\mathcal{L}^{\delta,\overline{m}_n^\delta}$ is a nondecreasing process, $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta>0}$ is P-UT as soon as $(Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1)_{\delta>0}$ is tight, where $Var(X)$ is the variation of the process X, cf. [30, Chapter VI, 6.6]. Note $Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1 = \mathcal{L}_1^{\delta,\overline{m}_n^\delta}$, since $\mathcal{L}^{\delta,\overline{m}_n^\delta}$ is nondecreasing. Then the tightness of $(Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1)_{\delta>0}$ is implied by Proposition 2.6.3. $\square$

Corollary 2.6.6 *For any $n \in \{1, \cdots, N\}$ and $t \in [0,1]$,*

$$\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}\left[\mathcal{L}_t^{\delta,\overline{m}_n^\delta}\right] = \lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}\left[\mathcal{L}_t^{\delta,\underline{m}_n^\delta}\right] = \mathbb{E}^{0,0}\left[\mathcal{L}_t^{m_n}\right].$$

Proof. For simplicity of presentation, we will prove $\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[\mathcal{L}_t^{\delta,0}] = \mathbb{E}^{0,0}[\mathcal{L}_t^0]$. Then the statement of the corollary follows from replacing Y_t^δ by $Y_t^\delta - \overline{m}_n^\delta$ or $Y_t^\delta - \underline{m}_n^\delta$ in the rest of the proof. Since the stochastic integrals in (2.6.10) are $\mathbb{P}^{\delta,0}$-martingales,

$$2\mathbb{E}^{\delta,0}[\mathcal{L}_t^{\delta,0}] = \mathbb{E}^{\delta,0}[|Y_t^\delta|].$$

Since $\mathbb{E}[(Y_t^\delta)^2] = t$ for any $\delta > 0$, $(|Y_t^\delta|; \mathbb{P}^{\delta,0})_{\delta>0}$ is uniformly integrable. It then follows from [24, Appendix, Proposition 2.3] and $Law(|Y_t^\delta|) \Longrightarrow Law(|Y_t^0|)$ that $\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[|Y_t^\delta|] = \mathbb{E}^{0,0}[|Y_t^0|]$. Therefore the claim follows since $\mathbb{E}^{0,0}[|Y_t^0|] = 2\mathbb{E}^{0,0}[\mathcal{L}_t^0]$ cf. (2.6.11). $\square$

Collecting previous results, the following result confirms (2.6.5).

Proposition 2.6.7 *For the strategies $(X^{B,\delta}, X^{S,\delta}; \mathscr{F}^{I,\delta})_{\delta>0}$ constructed in section 2.5,*

$$\lim_{\delta\downarrow 0} L^\delta(v_n, 0, 0) = 0, \quad n \in \{1, \cdots, N\}.$$

Proof. Fix any $\epsilon \in (0,1)$. Corollary 2.6.5 implies that $Law(\overline{I}^{\delta,n}_{1-\epsilon}; \mathscr{F}^{Y,\delta}) \Longrightarrow Law(I^{0,n}_{1-\epsilon}; \mathscr{F}^0)$. Recall $Law(\tilde{v}^\delta) \Longrightarrow Law(\tilde{v})$ from Lemma 2.6.1. It then follows

$$Law\left(\overline{I}^{\delta,n}_{1-\epsilon}\,\mathbb{I}_{\{\tilde{v}^\delta=v_n\}}; \mathscr{F}^{Y,\delta}\right) \Longrightarrow Law\left(I^{0,n}_{1-\epsilon}\,\mathbb{I}_{\{\tilde{v}=v_n\}}; \mathscr{F}^0\right).$$

On the other hand, since N is finite, p^δ is bounded uniformly in δ. Then there exists constant C such that $|\overline{I}^{\delta,n}_{1-\epsilon}|\,\mathbb{I}_{\{\tilde{v}^\delta=v_n\}} \leq C\mathcal{L}^{\delta,\overline{m}^\delta_n}_{1-\epsilon}$, where the expectation of the upper bound converges, cf. Corollary 2.6.6. Therefore appealing to [24, Appendix Theorem 1.2] and utilizing $\lim_{\delta\downarrow 0}\mathbb{P}^\delta(\tilde{v}^\delta = v_n) = \mathbb{P}^0(\tilde{v} = v_n)$ from Lemma 2.6.1, we obtain

$$\begin{aligned}\mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_{1-\epsilon}\,|\,\tilde{v}^\delta = v_n\right] &= \frac{\mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_{1-\epsilon}\,\mathbb{I}_{\{\tilde{v}^\delta=v_n\}}\right]}{\mathbb{P}^\delta(\tilde{v}^\delta = v_n)} \\ \rightarrow \frac{\mathbb{E}^{0,0}\left[I^{0,n}_{1-\epsilon}\,\mathbb{I}_{\{\tilde{v}=v_n\}}\right]}{\mathbb{P}^0(\tilde{v} = v_n)} &= \mathbb{E}^{0,0}\left[I^{0,n}_{1-\epsilon}\,|\,\tilde{v} = v_n\right],\end{aligned} \tag{2.6.12}$$

as $\delta \downarrow 0$. On the other hand, since $\lim_{\delta\downarrow 0}\mathbb{P}^\delta(\tilde{v}^\delta) = \mathbb{P}^0(\tilde{v} = v_n) > 0$, there exists a constant C such that

$$\mathbb{E}^{\delta,0}\left[|\overline{I}^{\delta,n}_1 - \overline{I}^{\delta,n}_{1-\epsilon}|\,\Big|\,\tilde{v}^\delta = v_n\right] \leq C\,\mathbb{E}^{\delta,0}\left[\mathcal{L}^{\delta,\overline{m}^\delta_n}_1 - \mathcal{L}^{\delta,\overline{m}^\delta_n}_{1-\epsilon}\right] \rightarrow C\,\mathbb{E}^{0,0}\left[\mathcal{L}^{m_n}_1 - \mathcal{L}^{m_n}_{1-\epsilon}\right],$$

as $\delta \downarrow 0$, where the convergence follows from applying Corollary 2.6.6 twice. For the difference of Brownian local time, Lévy's result (cf. [31, Chapter 3, Theorem 6.17]) yields

$$\begin{aligned}&\mathbb{E}^{0,0}\left[\mathcal{L}^{m_n}_1 - \mathcal{L}^{m_n}_{1-\epsilon}\right] = \mathbb{E}^{0,-m_n}\left[\mathcal{L}^0_1 - \mathcal{L}^0_{1-\epsilon}\right] \\ &= \frac{1}{2}\mathbb{E}^{0,-m_n}\left[\sup_{r\leq 1} Y^0_r - \sup_{r\leq 1-\epsilon} Y^0_r\right] = \sqrt{\frac{2}{\pi}}(1-\sqrt{1-\epsilon}),\end{aligned}$$

where Y^0 is a $\mathbb{P}^0$-Brownian motion and $\mathbb{E}^{0,y}[\sup_{r\leq t} Y^0_r] = \sqrt{2t/\pi} + y$ is utilized to obtain the third identity. Now the previous two estimates combined yield

$$\limsup_{\delta\downarrow 0}\mathbb{E}^{\delta,0}\left[|\overline{I}^{\delta,n}_1 - \overline{I}^{\delta,n}_{1-\epsilon}|\,\Big|\,\tilde{v}^\delta = v_n\right] \leq C(1-\sqrt{1-\epsilon}), \qquad \text{for another constant } C. \tag{2.6.13}$$

Estimates in (2.6.12) and (2.6.13) also hold when $\overline{I}^{\delta,n}$ is replaced by $\underline{I}^{\delta,n}$. These estimates then yield

$$\begin{aligned}&\mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_1 - \underline{I}^{\delta,n}_1\,|\,\tilde{v}^\delta = v_n\right] \\ &\leq \mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_{1-\epsilon} - \underline{I}^{\delta,n}_{1-\epsilon}\,|\,\tilde{v}^\delta = v_n\right] + \mathbb{E}^{\delta,0}\left[|\overline{I}^{\delta,n}_1 - \overline{I}^{\delta,n}_{1-\epsilon}|\,\Big|\,\tilde{v}^\delta = v_n\right] \\ &\quad + \mathbb{E}^{\delta,0}\left[|\underline{I}^{\delta,n}_1 - \underline{I}^{\delta,n}_{1-\epsilon}|\,\Big|\,\tilde{v}^\delta = v_n\right].\end{aligned}$$

Sending $\delta \downarrow 0$ in the previous inequality, the first term on the right side vanishes in the limit, because both conditional expectations converge to the same limit, the limit superior of both second and third terms are less than $C(1-\sqrt{1-\epsilon})$. Now since ϵ is arbitrarily choose, sending $\epsilon \rightarrow 1$ yields $\limsup_{\delta\downarrow 0}\mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_1 - \underline{I}^{\delta,n}_1\,|\,\tilde{v}^\delta = v_n\right] \leq 0$. Similar argument leads to

$$\liminf_{\delta\downarrow 0}\mathbb{E}^{\delta,0}\left[\overline{I}^{\delta,n}_1 - \underline{I}^{\delta,n}_1\,|\,\tilde{v}^\delta = v_n\right] \geq 0,$$

which concludes the proof. □

Finally the proof of Theorem 2.2.12 is concluded.

Proof of Theorem 2.2.12. It remains to verify Definition 2.2.11 iii). Fix v_n and $(y,t)=(0,0)$ throughout the proof. We have seen from Proposition 2.4.4 that $V^\delta \le U^{S,\delta}$. On the other hand, Proposition 2.4.2 yields $\mathcal{J}(X^{B,\delta}, X^{S,\delta}) = U^\delta - L^\delta$. Therefore

$$\sup_{(X^B,X^S)\text{ admissible}} \mathcal{J}^\delta(X^B,X^S) - \mathcal{J}^\delta(X^{B,\delta},X^{S,\delta}) \le U^{S,\delta} - U^\delta + L^\delta.$$

Since $\lim_{\delta\downarrow 0} L^\delta = 0$ is proved in Proposition 2.6.7, it suffices to show $\lim_{\delta\downarrow 0} U^{S,\delta} - U^\delta = 0$. To this end, from the definition of $U^{S,\delta}$,

$$U^{S,\delta}(0,0) - U^\delta(0,0) = (U^\delta(-\delta,0) - U^\delta(0,0))\, \mathbb{I}_{\{0\le \underline{m}_n^\delta\}} = \delta(v_n - p^\delta(0,0))\mathbb{I}_{\{0\le \underline{m}_n^\delta\}}. \tag{2.6.14}$$

The second identity above follows from (2.4.12) which reads $U^\delta(y,t) - U^\delta(y-1,t) + \delta(v_n - p^\delta(y,t)) = 0$ for $y \le \underline{m}_n^\delta$ when the order size is δ. Therefore $\lim_{\delta\downarrow 0} U^{S,\delta} - U^\delta = 0$ is confirmed after sending $\delta \downarrow 0$ in (2.6.14). □

2.7 Appendix

2.7.1 Viscosity Solution

Proposition 2.3.1 will be proved in this section. To simplify notation, $\delta = 1$ and $\tilde{v} = v_n$ are fixed throughout this section. First let us recall the definition of (discontinuous) viscosity solution to (2.2.8). Given a locally bounded function[11] $v : \mathbb{Z}\times[0,1] \to \mathbb{R}$, its *upper-semicontinuous envelope* v^* and *lower-semicontinuous envelope* v_* are defined as

$$v^*(y,t) := \limsup_{t'\to t} v(y,t'), \quad v_*(y,t) := \liminf_{t'\to t} v(y,t'), \quad (y,t) \in \mathbb{Z}\times[0,1]. \tag{2.7.1}$$

Definition 2.7.1 Let $v : \mathbb{Z}\times[0,1]\to\mathbb{R}$ be locally bounded.

i) v is a *(discontinuous) viscosity subsolution* of (2.2.8) if

$$-\varphi_t(y,t) - H(y,t,v^*) \le 0,$$

for all $y\in\mathbb{Z}$, $t\in[0,1)$, and any function $\varphi : \mathbb{Z}\times[0,1]\to\mathbb{R}$ continuously differentiable in the second variable such that (y,t) is a maximum point of $v^* - \varphi$.

[11] Since the state space $\mathbb{Z}$ is discrete, v is locally bounded if $v(y,\cdot)$ is bounded in any bounded neighborhood of t and any fixed $y\in\mathbb{Z}$.

ii) v is a *(discontinuous) viscosity supersolution* of (2.2.8) if

$$-\varphi_t(y,t) - H(y,t,v_*) \geq 0,$$

for all $y \in \mathbb{Z}$, $t \in [0,1)$, and any function $\varphi : \mathbb{Z} \times [0,1] \to \mathbb{R}$ continuously differentiable in the second variable such that (y,t) is a minimum point of $v_* - \varphi$.

iii) We say that v is a *(discontinuous) viscosity solution* of (2.2.8) if it is both subsolution and supersolution.

For the insider's optimization problem, let us recall the *dynamic programming principle* (cf. e.g. [38, Remark 3.3.3]). Given an admissible strategy (X^B, X^S), any $[t,1]-$valued stopping time τ, and the fundamental value v_n, denote the associated profit by

$$\begin{aligned}\mathcal{I}^n_{t,\tau} := &\int_t^\tau (v_n - p(Y_{r-}+1,r))dX_r^{B,B} + \int_t^\tau (v_n - p(Y_{r-}+2,r))dX_r^{B,T} \\ &+ \int_t^\tau (v_n - p(Y_{r-},r))dX_r^{B,S} - \int_t^\tau (v_n - p(Y_{r-}-1,r))dX_r^{S,S} \\ &- \int_t^\tau (v_n - p(Y_{r-}-2,r))dX_r^{S,T} - \int_t^\tau (v_n - p(Y_{r-},r))dX_r^{S,B},\end{aligned}$$

where $Y = Z + X^B - X^S$. Then the dynamic programming principle reads:

DPP i) For any admissible strategy (X^B, X^S) and any $[t,1]$-valued stopping time τ,

$$V(y,t) \geq \mathbb{E}^{y,t}[V(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau}].$$

DPP ii) For any $\epsilon > 0$, there exists an admissible strategy (X^B, X^S) such that for all $[t,1]$-valued stopping time τ,

$$V(y,t) - \epsilon \leq \mathbb{E}^{y,t}[V(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau}].$$

The viscosity solution property of the value function V follows from the dynamic programming principle and standard arguments in viscosity solutions, (see e.g. [38, Propositions 4.3.1 and 4.3.2][12].) Therefore Proposition 2.3.1 i) is verified.

Remark 2.7.2 The proof of DPP ii) utilizes the measurable selection theorem. To avoid this technical result, one could employ the *weak* dynamic programming principle in [12]. For the insider's optimization problem, the weak dynamic programming principle reads:

WDPP i) For any $[t,1]$-valued stopping time τ,

$$V(y,t) \leq \sup_{(X^B,X^S)} \mathbb{E}^{y,t}\left[V^*(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau}\right].$$

[12]Therein the stopping time τ_m can be chosen as the first jump time of Y where $Y_{t_m} = y$ for a sequence $(t_m)_m \to \bar{t}$

WDPP ii) For any $[t, 1]$-valued stopping time τ and any upper-semicontinuous function φ on $\mathbb{Z} \times [0, 1]$ such that $V \geq \varphi$, then

$$V(y, t) \geq \sup_{(X^B, X^S)} \mathbb{E}^{y,t} \left[\varphi(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau}\right].$$

Conditions A1, A2, and A3 from Assumption A in [12] are clearly satisfied in the current context. Condition A4 from Assumption A can be verified following the same argument in [12, Proposition 5.4]. Therefore aforementioned weak dynamic programming principle holds. Hence the value function is a viscosity solution to (2.2.8) following from arguments similar to [12, Section 5.2].

Now the proof of Proposition 2.3.1 ii) is presented. To prove $(v_n, y, t, V) \in dom(H)$, observe from the viscosity supersolution property of V that $H(v_n, y, t, V_*) < \infty$, hence $(v_n, y, t, V_*) \in \text{dom}(H)$. On the other hand, for any integrable intensities $\theta^{i,j}$, $i \in \{B, S\}$ and $j \in \{B, T, S\}$, due to Definition 2.2.2 iv), one can show $\mathbb{E}^{y,t}[\mathcal{I}^n_{t,1}]$ is a continuous function in t. As a supremum of a family of continuous function (cf. (2.2.7)), V is then lower-semicontinuous in t. Therefore $V_* \equiv V$, which implies $(v_n, y, t, V) \in \text{dom}(H)$ for any v_n, $(y, t) \in \mathbb{Z} \times [0, 1)$. It then follows from (2.3.1) and (2.3.2) that

$$V(y-1, t) + p(y-1, t) - v_n \leq V(y, t) \leq V(y-1, t) + p(y, t) - v_n, \text{ for any } (y, t) \in \mathbb{Z} \times [0, 1). \quad (2.7.2)$$

Taking limit supremum in t in the previous inequalities and utilizing the continuity of $t \mapsto p(y, t)$, it follows that the previous inequalities still hold when V is replaced by V^*, which means $(v_n, y, t, V^*) \in \text{dom}(H)$ for any v_n, $(y, t) \in \mathbb{Z} \times [0, 1)$. As a result, $H(v_n, y, t, V_*)$ and $H(v_n, y, t, V^*)$ have the reduced form (2.3.3) where V is replaced by V_* and V^*, respectively. Hence Definition 2.7.1 implies that V is a viscosity solution of (2.3.4).

To prove Proposition 2.3.1 iii) and iv), let us first derive a comparison result for (2.3.4). The function $v : \mathbb{Z} \times [0, 1] \to \mathbb{R}$ has at most polynomial growth in its first variable if there exist C and n such that $|v(y, t)| \leq C(1 + |y|^n)$, for any $(y, t) \in \mathbb{Z} \times [0, 1]$.

Lemma 2.7.3 *Assume that u (resp. v) has at most polynomial growth and that it is upper-semicontinuous viscosity subsolution (resp. lower-semicontinuous supersolution) to* (2.3.4). *If $u(\cdot, 1) \leq v(\cdot, 1)$, then $u \leq v$ in $\mathbb{Z} \times [0, 1)$.*

Assume this comparison result for a moment. Inequalities (2.7.2) and Assumption 2.2.5 combined imply that V is of at most polynomial growth. Then Lemma 2.7.3 and (2.7.1) combined yield $V_* \leq V^* \leq V_*$, which implies the continuity of $t \mapsto V(y, t)$, hence Proposition 2.3.1 iii) is verified. On the other hand, one can prove $\tilde{V}(y, t) := \mathbb{E}^{y,t}[V(Z_1, 1)]$ is of at most polynomial growth and is another viscosity solution to (2.3.4)[13]. Then Lemma 2.7.3 yields

$$V(y,t) = \tilde{V}(y,t) = \mathbb{E}^{y,t}\left[V(Z_1, 1)\right],$$

which confirms Proposition 2.3.1 iv) via the Markov property of Z.

Proof of Lemma 2.7.3. For $\lambda > 0$, define $\tilde{u} = e^{\lambda t} u$ and $\tilde{v} = e^{\lambda t} v$. One can check $\tilde{u}$ (resp. $\tilde{v}$) is a viscosity subsolution (resp. supersolution) to

$$-w_t + \lambda w - \left(w(y+1,t) - 2w(y,t) + w(y-1,t)\right)\beta = 0. \tag{2.7.3}$$

Since the comparison result for (2.7.3) implies the comparison result for (2.2.8), it suffices to consider u (resp. v) as the viscosity subsolution (resp. supersolution) of (2.7.3).

Let C and n be constants such that $|u|, |v| \le C(1 + |y|^n)$ on $\mathbb{Z} \times [0,1]$. Consider $\psi(y,t) = e^{-\alpha t}(y^{2n} + \tilde{C})$ for some constants α and $\tilde{C}$. It follows

$$\begin{aligned} -\psi_t + \lambda\psi + \left(\psi(y+1,t) - 2\psi(y,t) + \psi(y-1,t)\right)\beta \\ > e^{-\alpha t}\left((\alpha+\lambda)(y^{2n} + \tilde{C}) - 2\beta y^{2n}\right) > 0, \end{aligned}$$

when $\alpha + \lambda > 2\beta$. Choosing α satisfying the previous inequality, then $v + \xi\psi$, for any $\xi > 0$, is a viscosity supersolution to (2.7.3). Once we show $u \le v + \xi\psi$, the statement of the lemma then follows after sending $\xi \downarrow 0$.

Since both u and v have at most linear growth

$$\lim_{|y|\to\infty} (u - v - \xi\psi)(y,t) = -\infty. \tag{2.7.4}$$

Replacing v by $v+\xi\psi$, we can assume that u (resp. v) is a viscosity subsolution (resp. supersolution) to (2.7.3) and

$$\sup_{\mathbb{Z}\times[0,1]} (u - v) = \sup_{\mathcal{O}\times[0,1]} (u - v), \qquad \text{for some compact set } \mathcal{O} \subset \mathbb{Z}.$$

Then $u \le v$ follows from the standard argument in viscosity solutions (cf. e.g. [38, Theorem 4.4.4]), which we briefly recall below.

Assume $M := \sup_{\mathbb{Z}\times[0,1]}(u - v) = \sup_{\mathcal{O}\times[0,1]}(u - v) > 0$ and the maximum is attained at $(\overline{x}, \overline{t}) \in \mathcal{O} \times [0,1]$. For any $\epsilon > 0$, define

$$\Phi_\epsilon(x,y,t,s) := u(x,t) - v(y,s) - \phi_\epsilon(x,y,t,s),$$

where $\phi_\epsilon(x,y,t,s) := \frac{1}{\epsilon}[|x-y|^2 + |t-s|^2]$. The upper-semicontinuous function Φ_ϵ attains its maximum, denoted by M_ϵ, at $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$. One can show, using the same argument as in [38,

[13] Write $\tilde{V}(y,t) = \mathbb{E}^0\left[V(Z_{1-t} + y, 1)\right]$. One can utilize the Markov property of Z to show that $\tilde{V}$ is continuous differentiable and $\tilde{V}$ is a classical solution to (2.3.4).

Theorem 4.4.4],

$$M_\epsilon \to M \quad \text{and} \quad (x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \to (\overline{x}, \overline{x}, \overline{t}, \overline{t}) \in \mathcal{O}^2 \times [0,1]^2 \quad \text{as } \epsilon \downarrow 0.$$

Here $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \in \mathcal{O}^2 \times [0,1]^2$ for sufficiently small ϵ. Now observe that

- (x_ϵ, t_ϵ) is a local maximum of $(x,t) \mapsto u(x,t) - \phi_\epsilon(x, y_\epsilon, t, s_\epsilon)$;
- (y_ϵ, s_ϵ) is a local minimum of $(y,t) \mapsto v(y,s) + \phi_\epsilon(x_\epsilon, y, t_\epsilon, s)$.

Then the viscosity subsolution property of u and the supersolution property of v imply, respectively,

$$\begin{aligned} &-\frac{2}{\epsilon}(t_\epsilon - s_\epsilon) + \lambda u(x_\epsilon, t_\epsilon) - \left(u(x_\epsilon + 1, t_\epsilon) - 2u(x_\epsilon, t_\epsilon) + u(x_\epsilon, t_\epsilon)\right)\beta \le 0, \\ &-\frac{2}{\epsilon}(t_\epsilon - s_\epsilon) + \lambda v(y_\epsilon, s_\epsilon) - \left(u(y_\epsilon + 1, s_\epsilon) - 2v(y_\epsilon, s_\epsilon) + v(y_\epsilon, s_\epsilon)\right)\beta \ge 0. \end{aligned}$$

Taking difference of the previous inequalities yields

$$\begin{aligned} &(\lambda + 2\beta)(u(x_\epsilon, t_\epsilon) - v(y_\epsilon, s_\epsilon)) \\ &\qquad \le \beta\left(u(x_\epsilon + 1, t_\epsilon) + u(x_\epsilon - 1, t_\epsilon)\right) - \beta\left(v(y_\epsilon + 1, s_\epsilon) + v(y_\epsilon - 1, s_\epsilon)\right). \end{aligned}$$

Sending $\epsilon \downarrow 0$ on both sides, we obtain

$$\begin{aligned} (\lambda + 2\beta)M &= (\lambda + 2\beta)u(\overline{x}, \overline{t}) \\ &\le \beta\left(u(\overline{x}+1, \overline{t}) - v(\overline{x}+1, \overline{t})\right) + \beta\left(u(\overline{x}-1, \overline{t}) - v(\overline{x}-1, \overline{t})\right) \le 2\beta M, \end{aligned}$$

which contradicts with $\lambda M > 0$. □

Chapter 3

Monotone convergence of BSDEs driven by marked point processes and an application

3.1 Introduction

In Chapter 2, the value function of the insider presented in (2.2.7) has a boundary layer $V^{\delta}(\tilde{v}, y, 1) = \lim_{t \to 1} V^{\delta}(\tilde{v}, y, t)$. The boundary layer is due to the unbounded trading intensities of the insider. Where does this boundary layer comes from? To answer the question, we consider a family of control problems where insider's trading intensity is at most $n \in \mathbb{N}$. For each constrained value function, we represent it via solutions of a BSDE driven by a marked point process. The value function of these problems increasingly converges to the original value function as $n \to \infty$, which implies there also has a convergence in BSDEs.

To take the monotone limit, we extend Peng's [37] monotone convergence of BSDEs from Brownian setting to market point processes. Before considering the monotone convergence, we use [20] to prove the well-posedness of a family of penalised BSDEs.

Confortola and Fuhrman [20] studied a class of BSDEs driven by marked point processes. They prove existence, uniqueness, a priori estimates and continuous dependence upon data of the BSDE. They also apply this family of BSDEs to study a class on non-Markovian optimal control problems whose randomness is driven by marked point processes. In the Brownian setting, Peng [37] proved a monotone convergence result for supersolutions to BSDEs. More precisely, if there exists a sequence of càdlàg supersolutions Y^n (cf. Definition 1.2.1) converging to a process Y with $\mathbb{E}\Big[\sup_{0 \le t \le T} |Y_t|^2\Big] < \infty$, the process Y is also a càdlàg supersolution of the same BSDE. In addition, Peng considers a family of penalised BSDEs converging to the smallest supersolution satisfying the constrained BSDE.

First we consider a sequence of BSDEs

$$Y_t^n = \xi^n + \int_t^T g_s(Y_s^n, Z_s^n(\cdot))dA_s - \int_t^T \int_E Z_s^n(y)q(ds\,dy) + C_T^n - C_t^n, \tag{3.1.1}$$

for $0 \leq t \leq T$ and $n \in \mathbb{N}$, where g is Lipschitz continuous and C^n is a continuous increasing process representing penalty of violation of a constraint. Rewriting (3.1.1) as a Lipschitz BSDE, we use [20] to establish existence and uniqueness of a solution to (3.1.1). Next we establish a comparison theorem in Theorem 3.3.7 to show that the sequence $(Y^n)_{n\in\mathbb{N}}$ is increasing. If the sequence of supersolutions Y^n increasingly converges to the process Y in (3.1.1) with $\mathbb{E}\Big[\sup_{0\leq s\leq T}|Y_s|^2\Big] < \infty$, we show that the process Y is also a supersolution. Moreover, there exist C which is the weak limit of C^n and Z which is the strong limit of Z^n. Both Y and Z satisfy the BSDE with the constraint. Finally the general result described above is applied to the insider's optimisation problem to give a stochastic representation of the boundary layer.

This chapter is organised as follows. The next section introduces a market point process and defines spaces of functions, random variables and processes. In section 3.3, we prove the well-posedness of a BSDE and provides a comparison theorem which is used in the following sections. In section 3.4 we provide a monotone convergence theorem in Theorem 3.4.2 and consider the smallest supersolution subject to a given constraint on (Y, Z). Finally, in section 3.5, we consider an application, insider trading problem and represent the value function by a monotone sequence of BSDE solutions.

3.2 Marked point process

In this section, following [13, Chapter VIII], we introduce marked point processes and stochastic integrals with respect to them.

3.2.1 Definitions

Consider a measurable space $(E, \mathscr{E})$, and a random sequence $(T_n, \zeta_n)_{n\geq 0} \in [0,\infty) \times E$, where $(T_n)_{n\geq 0}$, starting from $T_0 = 0$, is an increasing sequence of non-anticipating random times to describe the occurrence of events and $\zeta_n \in E$ is a quantity observed at time T_n. We assume that T_n is non-explosive, i.e. $T_n \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$, which guarantees the number of events occurring on any finite interval, is almost surely finite. The random sequence $(T_n, \zeta_n)_{n\geq 0}$ is called a *marked point process*, where $(T_n)_{n\geq 0}$ is a point process and $(\zeta_n)_{n\geq 0}$ are marks. Define a *counting process* $N_t(K)$ by

$$N_t(K) = \sum_{n\geq 1} \mathbb{I}_{\{T_n \leq t\}} \mathbb{I}_{\{\zeta_n \in K\}}, \quad K \in \mathscr{E}. \tag{3.2.1}$$

We associate to each $K \in \mathscr{E}$ the *counting measure* μ such that

$$\mu((0,t], K) = N_t(K), \quad t \geq 0. \tag{3.2.2}$$

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space and satisfying the usual conditions. Denote by $\mathscr{P}$ the $\mathscr{F}$-predictable algebra on $\Omega \times [0,T]$. If any process H is $\mathscr{P} \otimes \mathscr{E}$-measurable satisfying

$$\mathbb{E}\Big[\int_0^T \int_E |H_t(k)|\, \mu(dt\ dk)\Big] < \infty, \tag{3.2.3}$$

it follows from [13, Chapter VIII, T14] that there exists a function ϕ_t and an increasing process A with $A_0 = 0$, such that

i) $K \to \phi_t(K)$ is a probability measure on $(E, \mathscr{E})$;

ii) $t \to \phi_t(K)$ is a predictable process;

iii) we have

$$\mathbb{E}\Big[\int_0^T \int_E H_t(k)\mu(dt\ dk)\Big] = \mathbb{E}\Big[\int_0^T \int_E H_t(k)\phi_t(dk)dA_t\Big]. \tag{3.2.4}$$

The predictable random measure $\phi_t(dk)dA_t$ is denoted by $\nu(dt, dk)$ and called the compensator of μ or dual predictable projection of μ. For H satisfies (3.2.3), we can define the compensated stochastic integral

$$\mathcal{M}_t := \int_0^t \int_E H_r(k) q(dr\ dk), \tag{3.2.5}$$

where $q(dt\, dk) := \mu(dt\, dk) - \nu(dt\, dk)$ is called the *compensated measure*. It follows from [13, Chapter VIII, C4] that $\mathcal{M}$ is a martingale.

3.2.2 Spaces of functions, random variables and stochastic processes

Assumption 3.2.1 The process A is an absolutely continuous increasing process with respect to time. There exists a constant α such that $A_T \leq \alpha$ a.s..

Now let us introduce the following spaces of functions, random variables and stochastic processes.

- Let $L^2(E; \mathbb{R})$ denote the space of $\mathscr{E}$-measurable functions φ: $E \to \mathbb{R}$ satisfying

$$\int_E |\varphi(y)|^2\, \phi_t(dy) < \infty;$$

- Let $\mathbb{L}^2(\Omega; \mathbb{R})$ denote the space of random variables ξ: $\Omega \to \mathbb{R}$ satisfying

$$|\xi|^2 := \mathbb{E}\Big[\, |\xi|^2 \Big] < \infty;$$

- Let $\mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ denote the space of adapted càdlàg processes $Y\colon \Omega \times [0,T] \to \mathbb{R}$, which are $\mathscr{G}$-measurable, satisfying

$$|Y|^2 := \mathbb{E}\Big[\int_0^T |Y_s|^2\, dA_s\Big] < \infty,$$

where $\mathscr{G}$ denoted by the $(\mathscr{F}_t)_{t\geq 0}$-progressive algebra on $\Omega \times [0,T]$;

- Let $\mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E;\mathbb{R})$ denote the space of processes $Z\colon \Omega \times [0,T] \times E \to \mathbb{R}$, which are $\mathscr{P} \otimes \mathscr{E}$-measurable, satisfying

$$||Z||^2 := \mathbb{E}\Big[\int_0^T \int_E |Z_s(y)|^2\, \phi_s(dy) dA_s\Big] < \infty;$$

- Let $\mathbb{S}^2_{\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ denote the space of adapted càdlàg processes $Y : \Omega \times [0,T] \to \mathbb{R}$, which are $\mathscr{G}$-measurable, satisfying

$$|Y|^2_{\sup} := \mathbb{E}\Big[\sup_{0\leq s\leq T} |Y_s|^2\Big] < \infty;$$

- Let $\mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ denote the subspace of $\mathbb{S}^2_{\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ which consists processes with non-decreasing trajectories;

- Let $\mathbb{S}^2_{inc,c,\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ denote the subspace of $\mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ which consists processes with continuous and non-decreasing trajectories;

To simplify notations, let $\mathbb{K}^2(\mathbb{R})$ denote the space of $\mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,T];\mathbb{R}) \times \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E;\mathbb{R})$, endowed with the norm $||(Y,Z)||^2 := |Y|^2 + ||Z||^2$.

3.3 Backward stochastic differential equation

In this section, we will consider a BSDE driven by a marked point process. Under suitable assumptions, we prove the existence and uniqueness of a solution. Finally we state a comparison principle which allows us to compare solutions of BSDEs.

3.3.1 Uniqueness and existence of the BSDE solution

Let us consider a pair of processes $(Y,Z) \in \mathbb{K}^2(\mathbb{R})$ satisfying

$$Y_t = \xi + \int_t^T g_s(Y_s,\, Z_s(\cdot))dA_s - \int_t^T \int_E Z_s(y) q(ds\, dy) + C_T - C_t, \quad 0 \leq t \leq T, \tag{3.3.1}$$

which is called the backward stochastic differential equation. In the above equation (3.3.1), the function g is called a *generator*, and the process Z is called *control process* as it controls an adapted process Y so that Y satisfies the terminal condition $Y_T = \xi$.

Definition 3.3.1 Given a non-decreasing process C, a terminal value ξ and a generator g, if the pair (Y, Z) solves (3.3.1), then we call Y a *supersolution* of BSDE with generator g or simply called *g-supersolution* on $[0, T]$. In particular, when $C \equiv 0$, Y is called a *g-solution* on $[0, T]$.

Assumption 3.3.2 There exist the process C, the random variable ξ and the generator g satisfying following conditions:

1) $C \in \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$ with $C_0 = 0$;

2) ξ is a $\mathscr{F}_T$-measurable random variable and $\xi \in \mathbb{L}^2(\Omega; \mathbb{R})$;

3) for $t \in [0, T]$, $r \in \mathbb{R}$, a mapping $g_t(r, \cdot)$: $L^2(E; \mathbb{R}) \to \mathbb{R}$ is given, then

   i) for every $Z \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0, T] \times E; \mathbb{R})$, the mapping $(t, r) \to g_t(r, Z_t(\cdot))$ is $\mathscr{G} \otimes \mathscr{B}(\mathbb{R})$-measurable;

   ii) we have $\mathbb{E}\Big[\int_0^T g_s(0,0) dA_s\Big] < \infty$;

   iii) there exist $\gamma_1 \geq 0$ and $\gamma_1' \geq 0$ such that r, $r' \in \mathbb{R}$ and z, $z' \in L^2(E; \mathbb{R})$, we have

$$\left|g_t(r, z(\cdot)) - g_t(r', z'(\cdot))\right| \leq \gamma_1' \left|r - r'\right| + \gamma_1 \left(\int_E \left|z(y) - z'(y)\right|^2 \phi_t(dy)\right)^{\frac{1}{2}}. \tag{3.3.2}$$

Proposition 3.3.3 *When Assumption 3.3.2 holds, there exists a unique solution* $(Y, Z) \in \mathbb{K}^2(\mathbb{R})$ *solving the BSDE (3.3.1), and* $Y \in \mathbb{S}^2_{\mathscr{G}}(\Omega \times [0, T]; \mathbb{R})$.

Proof of Proposition 3.3.3. We introduce a fixed positive constant $\beta > \gamma_1^2 + 2\gamma_1'$ and denote $\mathbb{K}^{2,\beta}(\mathbb{R})$ by a space satisfying the norm of (Y, Z) such that

$$||(Y, Z)||^2_\beta := |Y|^2_\beta + ||Z||^2_\beta,$$

where

$$|Y|^2_\beta := \mathbb{E}\Big[\int_0^T e^{\beta A_s} |Y_s|^2 dA_s\Big] < \infty,$$
$$||Z||^2_\beta := \mathbb{E}\Big[\int_0^T e^{\beta A_s} \int_E |Z_s(y)|^2 \phi_s(dy) dA_s\Big] < \infty.$$

The weighted space $\mathbb{K}^{2,\beta}(\mathbb{R})$ is equivalent to the unweighted space $\mathbb{K}^2(\mathbb{R})$ defined in section 3.2.2 due to $A_T \leq \alpha$ in Assumption 3.2.1.

Next we define a process $\overline{Y} := Y + C$ and a random variable $\overline{\xi} := \xi + C_T$ to construct an equivalent BSDE

$$\overline{Y}_t = \overline{\xi} + \int_t^T g_s(\overline{Y}_s - C_s, Z_s(\cdot)) dA_s - \int_t^T \int_E Z_s(y) q(ds\, dy), \quad 0 \leq t \leq T. \tag{3.3.3}$$

Once we can show there exists a unique solution $(\overline{Y}, Z)$ solving (3.3.3), it is equivalent to show that there exists a unique solution (Y, Z) solving (3.3.1).

We also define a mapping $\Gamma : \mathbb{K}^{2,\beta}(\mathbb{R}) \to \mathbb{K}^{2,\beta}(\mathbb{R})$ satisfying $(\overline{Y}, Z) = \Gamma(\overline{U}, V)$ if $(\overline{Y}, Z)$ satisfies

$$\overline{Y}_t = \overline{\xi} + \int_t^T g_s(\overline{U}_s - C_s, V_s)dA_s - \int_t^T \int_E Z_s(y)q(ds\,dy), \quad 0 \le t \le T. \tag{3.3.4}$$

The Assumption 3.3.2 on the generator g implies that $\mathbb{E}\left[\int_0^T e^{\beta A_s} \left|g_s(\overline{U}_s - C_s, V_s)\right|^2 dA_s\right] < \infty$, so by [20, Lemma 3.3], there exists a unique solution $(\overline{Y}, Z) \in \mathbb{K}^{2,\beta}(\mathbb{R})$ to (3.3.4) and Γ is well-defined.

Then we apply [20, Theorem 3.4] to represent that there exists a unique solution solving (3.3.3). The sketch of the proof is shown as below. Let us $(\overline{U}^i, V^i)$, $i = 1, 2$, be elements of $\mathbb{K}^{2,\beta}(\mathbb{R})$ and let $(\overline{Y}^i, Z^i) = \Gamma(\overline{U}^i, V^i)$. Denote $\Delta\overline{Y} = \overline{Y}^1 - \overline{Y}^2$, $\Delta Z = Z^1 - Z^2$, $\Delta\overline{U} = \overline{U}^1 - \overline{U}^2$ and $\Delta V = V^1 - V^2$. Using the Lipschitz conditions of g, [20, Theorem 3.4] establishes that

$$\left(\beta - \frac{\gamma_1^2}{c_3} - c_4\gamma_1'\right) \left|\Delta\overline{Y}\right|_\beta^2 + ||\Delta Z||_\beta^2 \le c_3 \,||\Delta V||_\beta^2 + \frac{\gamma_1'}{c_4}\left|\Delta\overline{U}\right|_\beta^2,$$

for every $c_3, c_4 > 0$. By the assumption on β, i.e. $\beta > \gamma_1^2 + 2\gamma_1'$, it is possible to find $c_3 \in (0, 1)$ such that

$$\beta > \frac{\gamma_1^2}{c_3} + \frac{2\gamma_1'}{\sqrt{c_3}}$$

If $\gamma_1' = 0$ we see that Γ is a contraction on $\mathbb{K}^{2,\beta}(\mathbb{R})$ endowed with the equivalent norm $(\overline{Y}, Z) \to \left(\beta - \frac{\gamma_1^2}{c_3}\right)\left|\overline{Y}\right|_\beta^2 + ||Z||_\beta^2$. If $\gamma_1' > 0$ we choose $c_4 = 1/\sqrt{c_3}$ and obtain

$$\frac{\gamma_1'}{\sqrt{c_3}}\left|\Delta\overline{Y}\right|_\beta^2 + ||\Delta Z||_\beta^2 \le c_3\,||\Delta V||_\beta^2 + \gamma_1'\sqrt{c_3}\left|\Delta\overline{U}\right|_\beta^2 = c_3\left(\frac{\gamma_1'}{\sqrt{c_3}}\left|\Delta\overline{U}\right|_\beta^2 + ||\Delta V||_\beta^2\right),$$

so that Γ is a contraction on $\mathbb{K}^{2,\beta}(\mathbb{R})$ endowed with the equivalent norm $(\overline{Y}, Z) \to \frac{\gamma_1'}{\sqrt{c_3}}\left|\overline{Y}\right|_\beta^2 + ||Z||_\beta^2$. In all cases there exists a unique fixed point which is the required unique solution to the BSDE (3.3.3). □

Proposition 3.3.3 illustrates that a supersolution Y on $[0, T]$ is uniquely determined once the terminal value ξ, the increasing càdlàg process C and the generator function g are given. On the other hand, we can extend [37, Proposition 1.6] for (3.3.1) to show:

Lemma 3.3.4 *Given a supersolution Y on $[0, T]$, there exists a unique pair $(Z, C) \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0, T] \times E; \mathbb{R}) \times \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0, T]; \mathbb{R})$ with $C_0 = 0$.*

Proof. We suppose two triplets (Y, Z, C) and (Y, Z', C') simultaneously satisfying (3.3.1). Applying Itô formula to $|Y_t - Y_t|^2 \equiv 0$, we have

$$\int_0^T \int_E \left|Z_s(y) - Z_s'(y)\right|^2 \phi_s(dy)dA_s + \sum_{t<s\le T}\left(\Delta\left|C_s - C_s'\right|\right)^2$$
$$+ \int_0^T \int_E \left|Z_s(y) - Z_s'(y)\right|^2 q(ds\,dy) = 0.$$

Then taking expectation on both sides, we have

$$\mathbb{E}\left[\int_0^T \int_E \left|Z_s(y) - Z_s'(y)\right|^2 \phi_s(dy)dA_s\right] + \mathbb{E}\left[\sum_{0\le s\le T}\left(\Delta\left|C_s - C_s'\right|\right)^2\right] = 0,$$

which makes sure $Z \equiv Z'$ and $C \equiv C'$ a.s.. $\square$

Definition 3.3.5 If (Z, C) is uniquely determined by the supersolution Y, (Z, C) is called the unique *decomposition* of Y.

3.3.2 Comparison theorem

Now we have proved the uniqueness and existence of the solution to the BSDE (3.3.1) under Assumption 3.3.2. In the next section, we can go further to consider the convergence of BSDEs. Before moving on to the next section, we present Comparison Theorem. The Comparison Theorem in [22] is used to compare two supersolutions of BSDEs driven by Brownian motions. Here we can present the comparison theorem for supersolutions of BSDEs driven by a marked point process.

Assumption 3.3.6 For any $t \in [0, T]$, $r \in \mathbb{R}$ and z, $z' \in L^2(E; \mathbb{R})$, there exist two constants $c_1 \geq c_2 > -1$ such that we can find a measurable map ρ: $\Omega \times [0, T] \times E \times \mathbb{R}^3 \to [c_2, c_1]$ satisfying

$$g_t(r, z(\cdot)) - g_t(r, z'(\cdot)) \leq \int_E (z(y) - z'(y))\rho_t^{r,z,z'}(y)\phi_t(dy). \tag{3.3.5}$$

Theorem 3.3.7 *(Comparison Theorem)*
Given (ξ^i, g^i, C^i) satisfying Assumption 3.3.2, let $(Y^i, Z^i) \in \mathbb{K}^2(\mathbb{R})$, $i = 1, 2$, be the unique solution to the following BSDE

$$Y_t^i = \xi^i + \int_t^T g_s^i(Y_s^i, Z_s^i(\cdot))dA_s - \int_t^T \int_E Z_s^i(y)q(ds\,dy) + C_T^i - C_t^i, \quad 0 \leq t \leq T,$$

where g^2 also satisfies Assumption 3.3.6. Moreover we assume that

i) $\xi^2 - \xi^1 \geq 0$, *a.s.,*

ii) $g_t^2(Y_t^1, Z_t^1(\cdot)) - g_t^1(Y_t^1, Z_t^1(\cdot)) \geq 0$, *a.s., a.e.,*

iii) $C_t^2 - C_t^1$ *is a càdlàg increasing process,*

then we have $Y_t^2 \geq Y_t^1$ as for all $t \in [0, T]$.

Proof. The proof is motivated by [23, Proposition 4.1] with proper modifications. Let us denote $\Delta Y := Y^2 - Y^1$, $\Delta\xi := \xi^2 - \xi^1$, $\Delta Z := Z^2 - Z^1$, $\Delta g := g^2(Y^2, Z^2(\cdot)) - g^1(Y^1, Z^1(\cdot))$ and $\Delta C := C^2 - C^1$ such that

$$\Delta Y_t = \Delta\xi + \int_t^T \Delta g_s dA_s - \int_t^T \int_E \Delta Z_s(y)q(ds\,dy) + \Delta C_T - \Delta C_t, \quad 0 \leq t \leq T. \tag{3.3.6}$$

Let us define a process a by

$$a_t := \frac{g_t^2(Y_t^2, Z_t^2(\cdot)) - g_t^2(Y_t^1, Z_t^2(\cdot))}{\Delta Y_t} \mathbb{I}_{\{\Delta Y_t \neq 0\}}, \quad 0 \leq t \leq T.$$

Note that the process a are bounded a.s. since g^2 is Lipschitz continuous. Observe also that the process ΔK defined on $[0,T]$ by

$$\Delta K_t := \Delta C_t - \int_0^t \int_E \rho_s^{Y_s^1, Z_s^1, Z_s^2}(y)\Delta Z_s(y)\phi_s(dy)dA_s + \int_0^t \left[g_s^2(Y_s^1, Z_s^2(\cdot)) - g_s^1(Y_s^1, Z_s^1(\cdot))\right] dA_s$$

is a non-decreasing process since

$$\begin{aligned} g_t^2(Y_t^1, Z_t^2(\cdot)) - g_t^1(Y_t^1, Z_t^1(\cdot)) &\geq g_t^2(Y_t^1, Z_t^2(\cdot)) - g_t^2(Y_t^1, Z_t^1(\cdot)) \\ &\geq \int_E \rho_t^{Y_t^1, Z_t^1, Z_t^2}(y)\Delta Z_t(y)\phi_t(dy), \end{aligned}$$

where the first inequality follows the assumption *ii*) in this Theorem and the second inequality follows Assumption 3.3.6. With these notations, we rewrite (3.3.6) as

$$\begin{aligned} \Delta Y_t =& \Delta\xi + \int_t^T a_s \Delta Y_s dA_s - \int_t^T \int_E \Delta Z_s(y) q(ds\, dy) \\ &+ \int_t^T \int_E \rho_s^{Y_s^1, Z_s^1, Z_s^2}(y)\Delta Z_s(y)\phi_s(dy)dA_s + \Delta K_T - \Delta K_t. \end{aligned}$$

Consider the positive process Ξ which is the solution of the following SDE

$$d\Xi_t = \Xi_t \left(a_t dA_t + \int_E \rho_t^{Y_t^1, Z_t^1, Z_t^2}(y) q(dt\, dy) \right), \quad \Xi_0 = 1.$$

Notice that $\mathbb{E}\Big[\sup_{0\leq s\leq T} |\Xi_s|^2\Big] < \infty$ since a and ρ are bounded, and Ξ is positive since $\rho > -1$ in Assumption 3.3.6. Applying the product rule to $d(\Xi_t \Delta Y_t)$, we have

$$\begin{aligned} d(\Xi_t \Delta Y_t) = \Xi_t \int_E \Delta Z_t(y) q(dt\, dy) + \Xi_t \int_E \Delta Z_t(y) \rho_t^{Y_t^1, Z_t^1, Z_t^2}(y) q(dt\, dy) \\ + \Delta Y_t \Xi_t \int_E \rho_t^{Y_t^1, Z_t^1, Z_t^2}(y) q(dt\, dy) - \Xi_t d(\Delta K_t). \end{aligned}$$

Hence, the process $\Xi\Delta Y$ is a supermartingale because of $\Xi > 0$. Therefore

$$\Xi_t \Delta Y_t \geq \mathbb{E}\left[\Xi_T \Delta\xi | \mathscr{F}_t\right] \geq 0, \quad 0 \leq t \leq T,$$

makes sure $\Delta Y \geq 0$. □

3.4 Monotone convergence of supersolutions

3.4.1 Monotone convergence theorem

In this section, we want to extend Peng's monotonic limit theorem in [37] for BSDEs driven by marked point processes. In order to show the monotone theorem of g-supersolutions, we need to introduce a sequence of triplets (Y^n, Z^n, C^n) satisfying the assumption as below.

Assumption 3.4.1 The following conditions hold for all $n \in \mathbb{N}$.

i) $\xi^n \in \mathbb{L}^2(\Omega;\mathbb{R})$ is a $\mathscr{F}_T$-measurable random variable;

ii) Y^n increasingly converges to Y with $Y \in \mathbb{S}^2_{\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$;

iii) $C^n \in \mathbb{S}^2_{inc,c,\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$ with $C^n_0 = 0$.

Given C^n, ξ^n and g satisfying Assumption 3.3.2, Proposition 3.3.3 implies the existence of a unique solution (Y^n, Z^n) solving

$$Y^n_t = \xi^n + \int_t^T g_s(Y^n_s, Z^n_s(\cdot))dA_s - \int_t^T \int_E Z^n_s(y)q(ds\,dy) + C^n_T - C^n_t, \quad 0 \le t \le T, \quad n \in \mathbb{N}. \tag{3.4.1}$$

Theorem 3.4.2 *(Monotone convergence theorem)*
When item 3) in Assumption 3.3.2, 3.3.6 and Assumption 3.4.1 are satisfied,

i) there exists a constant L such that $|Y^n|^2_{\sup} + ||Z^n||^2 + |C^n|^2_{\sup} \le L$ for all n;

ii) there exists a unique decomposition $(Z, C) \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E;\mathbb{R}) \times \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T];\mathbb{R})$, such that

$$|Y - Y^n|^2 + ||Z - Z^n||^\alpha \to 0, \quad 1 \le \alpha < 2,$$

and C is the weak limit of C^n. Moreover, (Y, Z, C) is the weak limit of (Y^n, Z^n, C^n).

iii) (Y, Z) is the solution of the BSDE

$$Y_t = \xi + \int_t^T g_s(Y_s,\, Z_s(\cdot))dA_s - \int_t^T \int_E Z_s(y)q(ds\,dy) + C_T - C_t, \quad 0 \le t \le T. \tag{3.4.2}$$

The proof of Theorem 3.4.2 is postponed to the Appendix.

3.4.2 BSDE with constraints

Now we will study the BSDE of type (3.3.1) with a constraint imposed to the solution, i.e.

$$Y_t = \xi + \int_t^T g_s(Y_s,\, Z_s(\cdot))dA_s - \int_t^T \int_E Z_s(y)q(ds\,dy) + C_T - C_t, \quad 0 \le t \le T, \tag{3.4.3}$$

as well as the constraint

$$h_t(Y_t, Z_t(y)) = 0, \quad \text{as for all } (t, y) \in [0,T] \times E. \tag{3.4.4}$$

Definition 3.4.3 Given a constraint function h, a terminal value ξ and a generator g, if the (Y, Z, C) solves (3.4.3), then we call Y a *supersolution* of BSDE subject to the constraint (3.4.4).

Assumption 3.4.4 The constrained function $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ satisfies following conditions:

i) h is $\mathscr{F}$-predictable and non-negative;

ii) there exist $\gamma_2 \geq 0$ and $\gamma_2' \geq 0$ such that for any $t \in [0,T]$ and r, r', z, $z' \in \mathbb{R}$, we have

$$\left|h_t(r,z) - h_t(r',z')\right| \leq \gamma_2' \left|r - r'\right| + \gamma_2 \left|z - z'\right|; \tag{3.4.5}$$

iii) for any $t \in [0,T]$, z, $z' \in \mathbb{R}$ and $z \geq z'$, $h_t(\cdot, z) \geq h_t(\cdot, z')$;

iv) $\mathbb{E}\left[\int_0^T |h_s(0,0)|^2 \, dA_s\right] < \infty$.

Assumption 3.4.5 We assume that there exists at least one g-supersolution $\widehat{Y} \in \mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$ solving (3.4.3)-(3.4.4) with unique decomposition $(\widehat{Z}, \widehat{C}) \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E; \mathbb{R}) \times \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$.

We will find the minimal g-supersolution later.

Definition 3.4.6 A g-supersolution Y on $[0,T]$ with the decomposition (Z,C) regarded as *the smallest g-supersolution*, given $Y_T = \xi$ subject to the constraint (3.4.4), whenever $Y \leq \widehat{Y}$ a.e., a.s., for any other g-supersolution $\widehat{Y}$ satisfying (3.4.3) and (3.4.4).

Theorem 3.4.7 *When Assumptions 3.3.2, 3.3.6, 3.4.1, 3.4.4 and 3.4.5 hold along with a sequence of BSDEs*

$$Y^n_t = \xi + \int_t^T g_s(Y^n_s, Z^n_s(\cdot)) dA_s - \int_t^T \int_K Z^n_s(y) q(ds\, dy) + C^n_T - C^n_t, \tag{3.4.6}$$

where

$$C^n_t := n \int_0^t \int_E h_s(Y^n_s, Z^n_s(y)) \phi_s(dy) dA_s. \tag{3.4.7}$$

then

i) there exists the smallest g-supersolution $Y \in \mathbb{S}^2_{\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$ to (3.4.3)-(3.4.4);

ii) the sequence Y^n increasingly converges to Y;

iii) there exists a unique decomposition $(Z,C) \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E; \mathbb{R}) \times \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$, such that

$$|Y - Y^n|^2 + ||Z - Z^n||^\alpha \to 0, \ \textit{for each } 1 \leq \alpha < 2,$$

and C is the weak limit of C^n. Moreover, (Y,Z,C) is the weak limit of (Y^n, Z^n, C^n).

Proof. Let us first prove the existence of the unique solution to (3.4.6). Define $f^n := g + nh$ and rearrange (3.4.6) to have

$$\begin{aligned} Y^n_t &= \xi + \int_t^T \left(g_s(Y^n_s, Z^n_s(\cdot)) + n \int_E h_s(Y^n_s, Z^n_s(y)) \phi_s(dy)\right) dA_s - \int_t^T \int_K Z^n_s(y) q(ds\, dy) \\ &= \xi + \int_t^T (g_s + nh_s)(Y^n_s, Z^n_s(\cdot)) dA_s - \int_t^T \int_K Z^n_s(y) q(ds\, dy) \\ &= \xi + \int_t^T f^n_s(Y^n_s, Z^n_s(\cdot)) dA_s - \int_t^T \int_K Z^n_s(y) q(ds\, dy). \end{aligned} \tag{3.4.8}$$

The Lipschitz property of f^n is satisfied since both g and h are Lipschitz. Moreover, for any $t \in [0,T]$, $r \in \mathbb{R}$ and z, $z' \in L^2(E;\mathbb{R})$, we have

$$\begin{aligned}
&f^n_t(r, z(\cdot)) - f^n_t(r, z'(\cdot)) \\
&\le \int_E (z(y) - z'(y))\rho^{r,z,z'}_t(y)\phi_t(dy) + n\int_E (h(r, z(y) - h(r, z'(y))\phi_t(dy) \\
&\le \int_E (z(y) - z'(y))\widehat{\rho}^{r,z,z'}_t(y)\phi_t(dy)
\end{aligned} \tag{3.4.9}$$

where $\widehat{\rho}^{r,z,z'}_t(\cdot) := \rho^{r,z,z'}_t(\cdot) + n\gamma_2\mathbb{I}_{\{z(\cdot)>z'(\cdot)\}}$ is located in $[c_2, c_1 + n\gamma_2]$ where $c_2 > -1$. The first inequality follows (3.3.5) and the second inequality follows the non-decreasing property of h in z (cf. Assumption 3.4.4 iii)). Then applying Proposition 3.3.3 by assuming $C \equiv 0$, it shows that there exists a unique solution (Y^n, Z^n) to (3.4.8), which implies the existence of the unique solution to (3.4.6).

Now let us prove that $(Y^n)_n$ is increasing by Comparison Theorem 3.3.7. Consider (Y^n, Z^n) as a solution to (3.4.8). Items *i)* - *iii)* of Theorem 3.3.7 are obviously satisfied. Moreover f^n satisfies (3.4.9) and f^n is increasing. Hence Theorem 3.3.7 implies that $(Y^n)_n$ is non-decreasing. For the g-supersolution $(\widehat{Y}, \widehat{Z}, \widehat{C})$ in Assumption 3.4.5, consider (Y^n, Z^n) as a solution to (3.4.8). Since $(\widehat{Y}, \widehat{Z}, \widehat{C})$ satisfies the constraint (3.4.4), $(\widehat{Y}, \widehat{Z}, \widehat{C})$ is a supersolution to (3.4.8). Hence Theorem 3.3.7 implies $Y^n \le \widehat{Y}$ for all n.

To prove convergence of (Y^n, Z^n, C^n), we directly apply Theorem 3.4.2. In addition since there exists a constant L such that $|C^n|^2_{\sup} \le L$, i.e.

$$\mathbb{E}\left[\sup_{0\le s\le T} |C^n_s|^2\right] = n^2\mathbb{E}\left[\sup_{0\le s\le T}\left(\int_0^t\int_E h_t(Y^n_s, Z^n_s(y))\phi_s(dy)dA_s\right)^2\right] \le L,$$

the constraint (3.4.4) is satisfied. In conclusion, (Y, Z, C) is one of the candidate solutions to (3.4.3)-(3.4.4).

Since the sequence of Y^n is upper-bounded by $\widehat{Y}$, i.e. $Y^n \le \widehat{Y}$, we take the limit on both sides to obtain

$$\lim_{n\uparrow\infty} Y^n = Y \le \widehat{Y},$$

which illustrates Y with decomposition (Z, C) is the minimal solution. Finally, by applying Lemma 3.3.4, we can state that there exists a unique decomposition (Z, C) of Y. $\square$

3.5 Application in an insider trading problem

3.5.1 Glosten-Milgrom model

Here we follow the assumptions and definitions in the last chapter. We consider a continuous time market for the risky asset whose *fundamental value* $\tilde{v}$ follows a discrete distribution,

$$\mathbb{P}(\tilde{v} = v_i) = p_i, \quad i = 1, \cdots, N, \tag{3.5.1}$$

where $N \in \mathbb{N} \cup \{\infty\}$, $(v_i)_{i=1,\cdots,N}$ is an increasing sequence and $p_i \in (0,1)$ with $\sum_{n=1}^{N} p_i = 1$. The risk free interest rate is 0. Without loss of generality, we assume the trading period is $[0,1]$ which can be scaled to any time interval. All of participants are risk neutral and will know this fundamental value at terminal time 1, at which point the market will terminate.

Noisy traders trade for liquidity reasons, and their aggregated demand is the difference of two pure jump processes Γ^B and Γ^S, which represents the cumulative buy and sell orders respectively. Noisy traders only submit orders with fixed size δ. Γ^B/δ and Γ^S/δ are assumed to be two independent Poisson processes with intensity λ. The net demand from noisy traders is defined as $\Gamma := \Gamma^B - \Gamma^S$. Then $(\mathscr{F}_t^{\Gamma})_{t\in[0,1]}$ describes the information structure of noise traders.

The *insider* observes the market price and knows the fundamental value $\tilde{v}$. The net order from the insider is defined as $I := I^B - I^S$. The admissibility of insider's strategy is defined in Definition 2.2.2. The insider's filtration $\mathscr{F}_t^I$ includes $\mathscr{F}_t^{\Gamma}$ and $\sigma(\tilde{v})$, for any $t \in [0,1]$. Moreover, the insider's buy orders I^B consist two components: we denote by $I^{B,B}$ the cumulative buy orders which arrive at different time than those of Γ^B, and by $I^{B,S}$ the cumulative buy orders which cancel some of Γ^S. Sell orders $I^{S,S}$ and $I^{S,B}$ are defined analogously. According to Definition 2.2.2 (iii) together with [30, Chapter1, Theorem 3.15], we have that $I_t^{i,j} - \delta\lambda \int_0^t \theta_s^{i,j} ds$ is a $\mathscr{F}^I$-martingale for $i \in \{B, S\}$ and $j \in \{B, S\}$ for any $t \in [0,1]$, where $\lambda\theta^{i,j}$ is $\mathscr{F}^I$-intensity.

A competitive *market maker* only observes the aggregated demand $M := I + \Gamma$ and sets the price only based on it. The market maker's filtration $\mathscr{F}^M$ is the smallest σ-field generated by M. Moreover, the net buy orders consists three components: $M^B := I^{B,B} + \Gamma^B - I^{S,B}$. The net sell orders $M^S := I^{S,S} + \Gamma^S - I^{B,S}$ is defined analogously. In addition, $M^B - \lambda\delta \int_0^{\cdot} \eta_s^B ds$ (resp. $M^S - \lambda\delta \int_0^{\cdot} \eta_s^S ds$) is a $\mathscr{F}^I$-martingale where $\eta^B := 1 + \theta^{B,B} - \theta^{S,B}$ (resp. $\eta^S := 1 + \theta^{S,S} - \theta^{B,S}$).

To simplify the presentation, we assume the order size $\delta = 1$. The definition of *pricing rule*, l, is the same as Definition 2.2.1. Given an admissible trading strategy $I := I^B - I^S$, we follow the similar way (cf. [18]) to derive the associated profit at time 1 for the insider conditional on her private information,

$$\mathbb{E}\left[\int_0^1 (\tilde{v} - a(M_{t-},t))\, dI_t^{B,B} + \int_0^1 (\tilde{v} - l(M_{t-},t))\, dI_t^{B,S} \right. \\ \left. - \int_0^1 (\tilde{v} - b(M_{t-},t))\, dI_t^{S,S} - \int_0^1 (\tilde{v} - l(M_{t-},t))\, dI_t^{S,B} \middle| \tilde{v} \right], \tag{3.5.2}$$

where $a(m,t) := l(m+1,t)$ is ask price and $b(m,t) := l(m-1,t)$ is bid price for any $m \in \mathbb{Z}$. Since the insider always aims to maximize her expected profit and $I^{i,j} - \delta\lambda \int_0^\cdot \theta_s^{i,j} ds$ is a $\mathscr{F}^I$-martingale for $i, j \in \{B, S\}$, the insider's value function can be expressed such as

$$V(\tilde{v},m,t) = \sup_{\eta^B,\eta^S \geq 0} \lambda\mathbb{E}\left[\int_t^1 (\tilde{v} - l(M_{s-}+1,s))(\eta_s^B - 1)^+ ds + \int_t^1 (\tilde{v} - l(M_{s-},s))(\eta_s^B - 1)^- ds \right. \\ \left. - \int_t^1 (\tilde{v} - l(M_{s-}-1,s))(\eta_s^S - 1)^+ ds - \int_t^1 (\tilde{v} - l(M_{t-},t))(\eta_s^S - 1)^- ds \middle| M_t = m, \tilde{v} \right], \tag{3.5.3}$$

for $\tilde{v} = \{v_1, \cdots, v_N\}$, $m \in \mathbb{Z}$ and $t \in [0,1)$. The terminal value of V is defined as $V(\tilde{v},m,1) = \lim_{t\uparrow 1} V(\tilde{v},m,t)$. Since η^B and η^S are uniquely determined by insider's strategy $(\theta^{i,j})$, we consider η^B and η^S as insider's control in (3.5.3).

Remark 3.5.1 The equation (3.5.3) is consistent with the value function (2.2.7), since the insider will hide her trades among noisy trades, i.e. not placing any buy (resp. sell) order to compensate a noisy buy (resp. sell) order. Therefore, second and fourth terms in (2.2.7) are equal to 0.

Instead of unbounded control in (3.5.3), we consider a family of control problems where the trading intensities η^B and η^S are bounded by n. The value function of the bounded control problem is

$$V^n(\tilde{v},m,t) = \sup_{\eta^B,\eta^S \in [0,n]} \lambda\mathbb{E}\left[\int_t^1 (\tilde{v} - l(M_{s-}+1,s))(\eta_s^B - 1)^+ ds + \int_t^1 (\tilde{v} - l(M_{s-},s))(\eta_s^B - 1)^- ds \right. \\ \left. - \int_t^1 (\tilde{v} - l(M_{s-}-1,s))(\eta_s^S - 1)^+ ds - \int_t^1 (\tilde{v} - l(M_{t-},t))(\eta_s^S - 1)^- ds \middle| M_t = m, \tilde{v} \right], \tag{3.5.4}$$

where the superscript n stands for the upper bound of trading intensities η^B and η^S. The terminal value $V^n(\tilde{v},m,1) = 0$. According to HJB equation (2.2.9), we can obtain HJB equation of V^n such that

$$\begin{aligned} &V_t^n(v_i,m,t) + \lambda\big[V^n(v_i,m+1,t) - 2V^n(v_i,m,t) + V^n(v_i,m-1,t)\big] \\ &+ (n-1)\big[(v_i - l(m+1,t)) + V^n(v_i,m+1,t) - V^n(v_i,m,t)\big]^+ \\ &+ (n-1)\big[(v_i - l(m-1,t)) + V^n(v_i,m,t) - V^n(v_i,m-1,t)\big]^- \\ &+ \big[v_i - l(m,t) + V^n(v_i,m,t) - V^n(v_i,m-1,t)\big]^+ \\ &+ \big[v_i - l(m,t) + V^n(v_i,m+1,t) - V^n(v_i,m,t)\big]^- = 0, \end{aligned} \tag{3.5.5}$$

where $i = \{1, \cdots, N\}$ and $(m, t) \in \mathbb{Z} \times [0, 1)$.

We notice that the terminal value of V in (3.5.3) is non-zero. This is so called "boundary layer" or "face-lifting" which usually appears when controls are unbounded. However, the terminal value of V^n is zero because the control is bounded. Our question is where this face-lifting comes from? Since the value functions V^n can be represented by the solution Y_n of a BSDE driven by a marked point process, the convergence of (Y_n) leads to a non-decreasing process C which we interpret as the boundary layer. In Lemma 3.5.2, by applying (4.17) and Theorem 4.10 in [20], we can find an expression of Y^n to represent the value functions V^n. Finally, in Proposition 3.5.3, we apply Theorem 3.4.7 to show the existence of the smallest supersolution Y which is exactly the value function V as n goes to infinity.

For any $n \in \mathbb{N}$, consider the following BSDE

$$Y_t^n = \int_t^1 g_s(Z_s^n(\cdot))ds - \int_t^1 \int_{\{-1,1\}} Z_s^n(y)q(ds\,dy) + C_1^n - C_t^n, \quad t \in [0, 1], \tag{3.5.6}$$

where

$$C_1^n - C_t^n = (n-1)\int_t^1 h_s(Z_s^n(\cdot))ds, \tag{3.5.7}$$

$$Z_t^n(y) = V^n(\tilde{v}, M_{t-} + y, t) - V^n(\tilde{v}, M_{t-}, t), \tag{3.5.8}$$

$$g_t(Z_t^n(\cdot)) = (\tilde{v} - l(M_{t-}, t) - Z_t^n(-1))^+ + (\tilde{v} - l(M_{t-}, t) + Z_t^n(1))^-, \tag{3.5.9}$$

$$h_t(Z_t^n(\cdot)) = (\tilde{v} - l(M_{t-} + 1, t) + Z_t^n(1))^+ + (\tilde{v} - l(M_{t-} - 1, t) - Z_t^n(-1))^-, \tag{3.5.10}$$

$$q(dt\,dy) = d(N_t(y) - 2\lambda t). \tag{3.5.11}$$

Here $N_t(y)$ is the counting process for $y \in \{-1, 1\}$ where 1 (resp. -1) represents a buy order (resp. sell order) to compensate the net order M.

Lemma 3.5.2 *The equation* (3.5.6) *admits a unique solution* (Y^n, Z^n) *and* $Y_t^n = V^n(\tilde{v}, M_t, t)$ *for all* $t \in [0, 1]$.

We put the proof of Lemma 3.5.2 in the Appendix. Now let us consider a scenario that the upper bound of the insider's intensity goes to infinity, i.e. $V^n = Y^n \uparrow V = Y$ as $n \uparrow \infty$. We first assume that the supersolution Y with a unique decomposition (Z, C) to satisfy

$$Y_t = \int_t^1 g_s(Z_s(\cdot))ds - \int_t^1 \int_{\{-1,1\}} Z_s(y)q(ds\,dy) + C_1 - C_t, \quad t \in [0, 1], \tag{3.5.12}$$

where $Y_1 = 0$, $Z_t(y) = V(\tilde{v}, M_{t-} + y, t) - V(\tilde{v}, M_{t-}, t)$, the definitions of the functions g and h are (3.5.9) and (3.5.10). Furthermore, there exists the inequality (2.2.10), for all $(m, t) \in \mathbb{Z} \times [0, 1)$ and $i = \{1, \ldots, N\}$, so we have

$$\begin{aligned} l(m,t) - v_i \le V(v_i, m+1, t) - V(v_i, m, t) \le l(m+1, t) - v_i, \\ v_i - l(m,t)- \le V(v_i, m-1, t) - V(v_i, m, t) \le v_i - l(m+1, t), \end{aligned} \tag{3.5.13}$$

which implies that the generator function $g_t(Z_t(\cdot)) = h_t(Z_t(\cdot)) \equiv 0$. When $t = 1$, we have the similar result. Hence the following supersolution Y of the BSDE to represent the insider's value function V such that

$$Y_t = -\int_t^1 \int_{\{-1,1\}} Z_s(y)q(ds\,dy) + C_1 - C_t, \quad t \in [0,1], \tag{3.5.14}$$

with the constraint

$$h_t(Z_t(\cdot)) = 0 \text{ a.e. a.s..}$$

In Proposition 3.5.3 below, we use Theorem 3.4.7 to show the convergence from (3.5.6) to (3.5.14) as $n \uparrow \infty$.

Proposition 3.5.3 *The sequence $(Y^n)_{n\in\mathbb{N}}$ in Lemma 3.5.2 increasingly converges to Y which is the smallest supersolution of* (3.5.14) *and*

$$|Y - Y^n|^2 + ||Z - Z^n||^\alpha \to 0, \quad 1 \leq \alpha < 2,$$

and C is the weak limit of C^n.

Proof. Let us check that all assumptions of Theorem 3.4.7 are satisfied. Then the statement readily follows from Theorem 3.4.7. Assumptions 3.3.2, 3.3.6 and 3.4.4 are automatically satisfied because of Lemma 3.5.2. Moreover Assumption 3.4.5 holds obviously. Now let us check Assumption 3.4.1. The item (i) is automatically satisfied as $Y_1^n = 0$. As Y^n represents the value function V^n when the intensities η^B and η^S are bounded by n, Y^n is increasing. In addition, according to (2.4.15), there exists a upper bound for the value function, such that

$$0 < V^n(v_i, m, t) \leq V(v_i, m, t) \leq U^S(v_i, m, t), \quad \text{for } m \in \mathbb{Z} \text{ and } i = \{1, \cdots, N\}.$$

Moreover, as the definition of U^S, we have $U^S(v_i, m, t) = \lambda \int_0^t (l(m,s) - l(m-1,s))ds$. Combining with $|v_i|^2 < \infty$ for $i = \{1, \cdots, N\}$, we confirm $U^S \in \mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,1]; \mathbb{R})$ which implies $Y \in \mathbb{S}^2_{\mathscr{G}}(\Omega \times [0,1]; \mathbb{R})$. We can also determine $C^n \in \mathbb{S}^2_{inc,c,\mathscr{G}}(\Omega \times [0,1]; \mathbb{R})$ with $C_0^n = 0$ by (3.5.7). Hence Assumption 3.4.1 is satisfied. Finally we apply Theorem 3.4.7 to obtain for $t \in [0,1]$,

$$\begin{aligned} Y_t &= \int_t^1 g_s(Z_s(\cdot))ds - \int_t^1 \int_{\{-1,1\}} Z_s(y)q(ds\,dy) + C_1 - C_t, \\ &= -\int_t^1 \int_{\{-1,1\}} Z_s(y)q(ds\,dy) + C_1 - C_t, \end{aligned} \tag{3.5.15}$$

with the constraint

$$h_t(Z_t(\cdot)) = 0 \text{ a.e. a.s..} \qquad \square$$

3.5.2 Numerical results

In the following numeric example, we investigate the convergence rate of V^n to V as $n \to \infty$. In this example, we assume $\tilde{v}$ follows Bernoulli distribution $\{0, 1\}$ with equal probabilities. We also assume that the intensity of noisy orders λ is 300 and the order size δ is defined as $1/\sqrt{2\lambda}$. Hence the value V is 0.3787 by applying equation (3.6) in [18] when insider's intensities η^B and η^S are unbounded. Next we also need to solve HJB equation (3.5.5) numerically to obtain V^n when the insider's intensities η^B and η^S are bounded by n. The Figure 3.1 shows that $V - V^n \sim n^{-0.32}$.

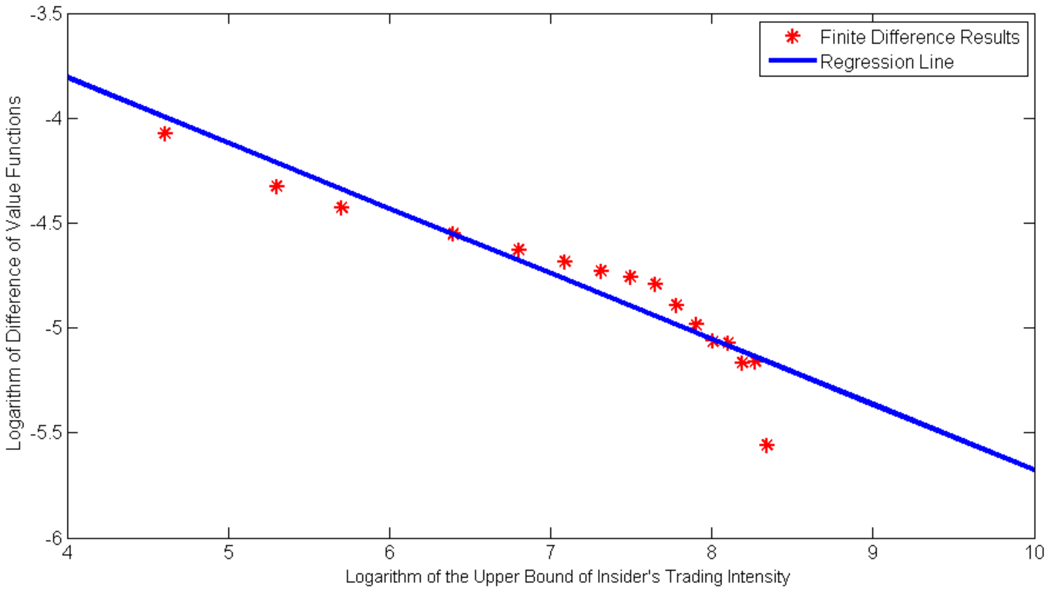


Figure 3.1: Simulation results and regression line for $(ln(n), ln(V - V^n))$. The slope of the regression line is about -0.32.

3.6 Appendix

Proof of Theorem 3.4.2. *1. Boundedness.* Since (Y^n) is monotonic, there exists a constant L' such that

$$\sup_{n\in\mathbb{N}} |Y^n|^2_{\text{sup}} \leq \left|Y^0\right|^2_{\text{sup}} + |Y|^2_{\text{sup}} \leq L', \tag{3.6.1}$$

where L' is independent of index n. Applying Itô formula to $|Y^n_t|^2$, recalling C^n is continuous and taking the expectation, we have

$$\begin{aligned} &\mathbb{E}\left[|Y^n_0|^2\right] + \mathbb{E}\left[\int_0^T \int_E |Z^n_s(y)|^2 \phi_s(dy) dA_s\right] \qquad (3.6.2)\\ &= \mathbb{E}\left[|\xi^n|^2\right] + 2\mathbb{E}\left[\int_0^T Y^n_s g^n_s(Y^n_s, Z_s(\cdot)) dA_s\right] + 2\mathbb{E}\left[\int_0^T Y^n_s dC^n_s\right] \\ &\leq \mathbb{E}\left[|\xi^n|^2\right] + 2\mathbb{E}\left[\int_0^T |Y^n_s| \left[|g_s(0,0)| + \gamma'_1 |Y^n_s| + \gamma_1\Big(\int_E |Z^n_s(y)|^2 \phi_s(dy)\Big)^{\frac{1}{2}}\right] dA_s\right] \end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\left[C_T^n \sup_{s\in[0,T]} |Y_s^n|\right] \\
&\le \mathbb{E}\left[|\xi^n|^2\right] + 2\mathbb{E}\int_0^T \left[(\gamma_1' + \gamma_1^2 + 1)\,|Y_s^n|^2 + \frac{1}{4}\int_E |Z_s^n(y)|^2\,\phi_s(dy) + \frac{1}{4}\,|g_s(0,0)|^2\right] dA_s \\
&+ 2\mathbb{E}\left[C_T^n \sup_{s\in[0,T]} |Y_s^n|\right],
\end{aligned}
$$

where the first inequality follows the Lipschitz condition of g. Then we rearrange the above inequality to have

$$
\begin{aligned}
||Z^n||^2 \le 2\mathbb{E}\Big[\,|\xi^n|^2\,\Big] + 4\mathbb{E}\left[C_T^n \sup_{s\in[0,T]} |Y_s^n|\right] \\
+ 4\mathbb{E}\left[\int_0^T \left[(\gamma_1' + \gamma_1^2 + 1)\,|Y_s^n|^2 + \frac{1}{4}\,|g_s(0,0)|^2\,\right] dA_s\right].
\end{aligned}
\tag{3.6.3}
$$

Since

$$
C_T^n \le |Y_0^n| + |\xi^n| + \int_0^T |g_s(Y_s^n,\, Z_s^n(\cdot))|\,dA_s + \int_0^T\int_E |Z_s^n(y)|\,q(ds\,dy),
$$

there exists a constant $\gamma_3 > 0$ such that

$$
|C^n|^2_{\text{sup}} \le \gamma_3\Big(1 + |Y^n|^2_{\text{sup}} + ||Z^n||^2\Big). \tag{3.6.4}
$$

Applying the inequality $4ab \le 8\gamma_3\,|a|^2 + \frac{|b|^2}{2\gamma_3}$ for $a,\, b \in \mathbb{R}$, we have

$$
\begin{aligned}
4\mathbb{E}\left[C_T^n \sup_{s\in[0,T]} |Y_s^n|\right] &\le 8\gamma_3\,|Y^n|^2_{\text{sup}} + \frac{|C^n|^2_{\text{sup}}}{2\gamma_3} \\
&\le 8\gamma_3\,|Y^n|^2_{\text{sup}} + \frac{1}{2}\Big(1 + |Y^n|^2_{\text{sup}} + ||Z^n||^2\Big) \\
&\le 8\gamma_3\,|Y^n|^2_{\text{sup}} + \frac{1}{2}\,||Z^n||^2 + \frac{1}{2}\,|Y^n|^2_{\text{sup}} + \frac{1}{2}.
\end{aligned}
$$

Combining the last inequality with (3.6.1) and (3.6.3), we obtain a constant γ_4 such that

$$
||Z^n||^2 + |Y^n|^2_{\text{sup}} \le \gamma_4.
$$

Then combining the last inequality with (3.6.4), there exists a constant L such that

$$
||Z^n||^2 + |Y^n|^2_{\text{sup}} + |C^n|^2_{\text{sup}} \le L.
$$

2. Weak convergence. According to the uniform boundedness, we need to show that there exists a subsequence of $(Z^n, C^n, g_\cdot(Y^n, Z^n))$ weakly converging to $(Z, C, G) \in \mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E; \mathbb{R}) \times \mathbb{S}^2_{inc,\mathscr{G}}(\Omega \times [0,T]; \mathbb{R}) \times \mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$. Identifying the limits of $(Y^n)_n$ and $(Z^n, C^n, g_\cdot(Y^n, Z^n))_n$, there exists a G such that

$$
Y_t = \xi + \int_t^T G_s dA_s - \int_t^T\int_E Z_s(y) q(ds\,dy) + C_T - C_t, \quad 0 \le t \le T. \tag{3.6.5}
$$

To prove the above identity, we first claim that the stochastic integral $\int_t^T \int_E Z_s^n(y)q(ds\,dy)$ converges weakly to $\int_t^T \int_E Z_s(y)q(ds\,dy)$. For any $\eta \in \mathbb{L}^2(\Omega;\mathbb{R})$, by martingale representation theorem, there exists a predictable process φ such that

$$\eta = \mathbb{E}\Big[\eta\Big] + \int_0^T \varphi_s(y)q(ds\,dy). \tag{3.6.6}$$

It implies a convergence such that

$$\begin{aligned}\mathbb{E}\left[\left[\int_t^T \int_E Z_s^n(y)q(ds\,dy), \eta\right]\right] &= \mathbb{E}\left[\int_t^T \int_E Z_s^n(y)\varphi_s(y)q(ds\,dy)\right] \\ \to \mathbb{E}\left[\int_t^T \int_E Z_s(y)\varphi_s(y)q(ds\,dy)\right] &= \mathbb{E}\left[\left[\int_t^T \int_E Z_s(y)q(ds\,dy), \eta\right]\right].\end{aligned}$$

Now define $I_{t,T}^n$ and $I_{t,T}$ such that

$$\begin{aligned}I_{t,T}^n &:= -Y_t^n + \xi + \int_t^T g_s(Y_s^n, Z_s^n(\cdot))dA_s - \int_t^T \int_E Z_s^n(y)q(ds\,dy) + C_T^n - C_t^n, \\ I_{t,T} &:= -Y_t + \xi + \int_t^T G_s dA_s - \int_t^T \int_E Z_s(y)q(ds\,dy) + C_T - C_t.\end{aligned}$$

Due to $I_{t,T}^n = 0$, the weak convergence of all terms on the right implies $I_{t,T} = 0$ a.s.. Therefore (3.6.5) is satisfied.

3. Properties of the process C. All of proofs in this part have been done by Peng in [37]. According to [37, Lemma 2.2], we know the process C is càdlàg. In addition, [37, Lemma A.1] tells us the process C has a finite number of jumps. We can construct a successive jump times $(\sigma_n)_{n=0}^{N+1}$ with $\sigma_0 = 0$, $\sigma_{N+1} = T$ and jump size bigger than a sufficiently small constant $\nu > 0$. Moreover, [37, Lemma 2.3] allows us to find another a sequence of jump times $(\tau_n)_{n=0}$ and construct a finite number of pairs of jump times $(\sigma_u, \tau_u)_{0 \le u \le N}$ with $0 \le \sigma_u \le \tau_u \le T$ such that

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset$ for each $j \neq k$;

(ii) $\mathbb{E}\left[\sum_{u=0}^N (\tau_u - \sigma_u)\right] \ge T - \epsilon$;

(iii) $\mathbb{E}\left[\sum_{u=0}^N \sum_{\sigma_u < t \le \tau_u} (\Delta C_t)^2\right] \le \delta$.

This result means that for any càdlàg increasing process defined on $[0, T]$, the total size of jumps in the process mainly concentrated within a finite number of time intervals is sufficiently small.

4. Strong convergence. Since $\left|Y_t^0 - Y_t\right|^2 \ge |Y_t^n - Y_t|^2 \to 0$ a.s. for all $t \in [0, T]$, and

$$\mathbb{E}\left[\int_0^T \left|Y_s^0 - Y_s\right|^2 dA_s\right] < \infty,$$

applying dominated convergence theorem, we have

$$|Y - Y^n|^2 \to 0, \text{ as } n \to \infty. \tag{3.6.7}$$

Now let us prove the strong convergence of Z. According to the statement of part 3, there exists a finite number of disjoint intervals $(\sigma_u, \tau_u]$, $u = 0, 1, \cdots, N$, satisfying

(i) $\mathbb{E}\left[\sum_{u=0}^{N}(\tau_u-\sigma_u)\right]\geq T-\frac{\epsilon}{2}$;

(ii) $\mathbb{E}\left[\sum_{u=0}^{N}\sum_{\sigma_u<t\leq\tau_u}(\Delta C_t)^2\right]\leq\frac{\delta\epsilon}{3}$.

Applying Itô formula to $|Y_t-Y_t^n|^2$ on $(\sigma_u,\tau_u]$ and summing over u, we have

$$\begin{aligned}
&\sum_{u=0}^{N}\mathbb{E}\left[\int_{\sigma_u}^{\tau_u}\int_E|Z_s^n(y)-Z_s(y)|^2\,\phi_s(dy)dA_s\right]+\sum_{u=0}^{N}\mathbb{E}\left[\left|Y_{\sigma_u}^n-Y_{\sigma_u}\right|^2\right]\\
&=\sum_{u=0}^{N}\mathbb{E}\left[\left|Y_{\tau_u}^n-Y_{\tau_u}\right|^2\right]+2\sum_{u=0}^{N}\mathbb{E}\left[\int_{\sigma_u}^{\tau_u}(Y_s^n-Y_s)(g_s(Y_s^n,Z_s^n(\cdot))-G_s)dA_s\right]\\
&\quad-\mathbb{E}\left[\sum_{u=0}^{N}\sum_{\sigma_u<s\leq\tau_u}|\Delta C_s|^2\right]-2\sum_{u=0}^{N}\mathbb{E}\left[\int_{\sigma_u}^{\tau_u}(Y_s^n-Y_s)dC_s\right]+2\sum_{u=0}^{N}\mathbb{E}\left[\int_{\sigma_u}^{\tau_u}(Y_s^n-Y_s)dC_s^n\right].
\end{aligned}$$

As the last term of the above identity are less than zero and $\mathbb{E}\left[\sum_{u=0}^{N}(\tau_u-\sigma_u)\right]\leq T$, we have

$$\begin{aligned}
&\sum_{u=0}^{N}\mathbb{E}\left[\int_{\sigma_u}^{\tau_u}\int_E|Z_s^n(y)-Z_s(y)|^2\,\phi_s(dy)dA_s\right] &(3.6.8)\\
&\leq\sum_{u=0}^{N}\mathbb{E}\left[\left|Y_{\tau_u}^n-Y_{\tau_u}\right|^2\right]+2\mathbb{E}\left[\int_0^T|Y_s^n-Y_s|\,|g_s(Y_s^n,Z_s^n(\cdot))-G_s|\,dA_s\right]\\
&\quad+\mathbb{E}\left[\sum_{u=0}^{N}\sum_{\sigma_u<s\leq\tau_u}|\Delta C_s|^2\right]+2\mathbb{E}\left[\int_0^T|Y_s^n-Y_s|\,dC_s\right].
\end{aligned}$$

As

$$N\sup_{s\in[0,T]}|Y_s^n-Y_s|^2\geq\sum_{n=0}^{N}\left|Y_{\tau_n}^n-Y_{\tau_n}\right|^2\to 0,$$

and

$$N\mathbb{E}\left[\sup_{s\in[0,T]}|Y_s^n-Y_s|^2\right]<\infty,$$

applying dominated convergence theorem, we have

$$\sum_{u=0}^{N}\mathbb{E}\left[\left|Y_{\tau_u}^n-Y_{\tau_u}\right|^2\right]\to 0,\ \text{as } n\to\infty. \tag{3.6.9}$$

Moreover, as $\left|Y_t^0-Y_t\right|\geq|Y_t^n-Y_t|\to 0$ for $t\in[0,T]$ and

$$\mathbb{E}\left[\int_0^T\left|Y_t^0-Y_t\right|dC_s\right]\leq\left(\mathbb{E}\left[\sup_{s\in[0,T]}\left|Y_s^0-Y_s\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|C_T|^2\right]\right)^{\frac{1}{2}}<\infty,$$

applying dominated convergence theorem, we have

$$\mathbb{E}\left[\int_0^T|Y_s^n-Y_s|\,dC_s\right]\to 0,\ \text{as } n\to\infty. \tag{3.6.10}$$

Finally, applying Cauchy Schwartz inequality, we have

$$\mathbb{E}\left[\int_0^T|Y_s^n-Y_s|\,|g_s(Y_s^n,Z_s^n(\cdot))-G_s|\,dA_s\right] \tag{3.6.11}$$

$$\leq \left(\mathbb{E} \left[\int_0^T |g_s(Y_s^n, Z_s^n(\cdot)) - G_s|^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_0^T |Y_s^n - Y_s|^2 \, dA_s \right] \right)^{\frac{1}{2}}$$
$$\leq L \left(\mathbb{E} \left[\int_0^T |Y_s^n - Y_s|^2 \, dA_s \right] \right)^{\frac{1}{2}} \to 0, \text{ as } n \to \infty,$$

where the second inequality holds since g satisfies Assumption 3.3.2 and $G \in \mathbb{L}^2_{\mathscr{G}}(\Omega \times [0,T]; \mathbb{R})$. Combining (3.6.8), (3.6.9), (3.6.10) and (3.6.11), we have

$$\overline{\lim_{n\uparrow\infty}} \sum_{u=0}^{N} \mathbb{E} \left[\int_{\sigma_u}^{\tau_u} \int_E |Z_s^n(y) - Z_s(y)|^2 \, \phi_s(dy) dA_s \right] \leq \mathbb{E} \left[\sum_{u=0}^{N} \sum_{\sigma_u < s \leq \tau_u} |\Delta C_s|^2 \right] \leq \frac{\delta\epsilon}{3}.$$

Thus there exists an integer $l_{\delta,\epsilon} > 0$ such that when $n > l_{\delta,\epsilon} \geq 0$,

$$\sum_{u=0}^{N} \mathbb{E} \left[\int_{\sigma_u}^{\tau_u} \int_E |Z_s^n(y) - Z_s(y)|^2 \, \phi_s(dy) dA_s \right] \leq \frac{\delta\epsilon}{2}.$$

Therefore, in the product space $(\Omega \times [0,T] \times E, \, \mathscr{P} \otimes \mathscr{E})$, for $n \geq l_{\delta,\epsilon}$, we have

$$\tilde{p} \otimes \mathbb{P} \left((s,y,\omega) \in \cup_{u=0}^{N} (\sigma_u, \tau_u] \times E \times \Omega; \quad |Z_s^n(y) - Z_s(y)|^2 \geq \delta \right) \leq \epsilon,$$

where $\tilde{p}$ is predictable random measure on $\mathscr{P} \otimes \mathscr{E}$. This implies that

$$\lim_{n\uparrow\infty} \tilde{p} \otimes \mathbb{P} \left((s,y,\omega) \in \cup_{u=0}^{N} (\sigma_u, \tau_u] \times E \times \Omega; \quad |Z_s^n(y) - Z_s(y)|^2 \geq \delta \right) = 0.$$

Thus Z^n converges in measure to Z. Since Z^n is bounded in $\mathbb{L}^2_{\mathscr{P}}(\Omega \times [0,T] \times E; \mathbb{R})$, then for each $\alpha \in [1,2)$, it converges in $\mathbb{L}^\alpha_{\mathscr{P}}(\Omega \times [0,T] \times E; \mathbb{R})$. Combining (3.6.7) and strong convergence of Z, we have

$$|Y - Y^n|^2 + ||Z - Z^n||^\alpha \to 0, \quad \text{for each } 1 \leq \alpha < 2.$$

Since we already know the strong convergence of (Y^n, Z^n) to (Y, Z), we know $g_t(Y_t^n, Z_t^n(\cdot))$ strongly converges to $g_t(Y_t, Z_t(\cdot))$ by showing

$$\mathbb{E} \left[\int_t^T |g_s(Y_s^n, Z_s^n(\cdot)) - g_s(Y_s, Z_s(\cdot))| \, dA_s \right]$$
$$\leq \mathbb{E} \left[\int_t^T \gamma_1' \, |Y_s^n - Y_s| \, dA_s \right] + \mathbb{E} \left[\int_t^T \gamma_1 \left(\int_E |Z_s^n(y) - Z_s(y)|^2 \, \phi_s(dy) \right)^{\frac{1}{2}} dA_s \right] \to 0.$$

Here $g_t(Y_t, Z_t(\cdot))$ identifies with G_t, i.e.

$$G_t = g_t(Y_t, Z_t(\cdot)), \quad 0 \leq t \leq T,$$

which immediately shows (Y, Z) is the solution of the BSDE (3.4.2). □

Proof of Lemma 3.5.2. Thanks to Theorem 4.11 in [20], we can apply it to prove this lemma. Before applying the theorem, we need to check whether we have satisfied Hypotheses 4.1 and 4.9

of [20]. In the insider problem, control space is a measurable space. When the controller η^i, $i \in \{B, S\}$, bounded by n, the profit of the insider is bounded. The value function $V^n(\tilde{v}, M_1, 1)$ is $\mathscr{F}_1^M$-measurable. In addition, for each of n, there exists an optimal strategy to maximise her expected profit. Therefore Hypotheses 4.1 and 4.9 in [20] are satisfied. Now we can apply Theorem 4.10 in [20] to generate a sequence of BSDEs

$$\begin{aligned} Y_t^n =&(n-1)\int_t^1 \Big[\big(\tilde{v} - l(M_{s-}+1, s) + Z_s^n(1)\big)^+ + \big(\tilde{v} - l(M_{s-}-1, s) - Z_s^n(-1)\big)^-\Big] ds \\ &+ \int_t^1 \Big[\big(\tilde{v} - l(M_{s-}, s) - Z_s^n(-1)\big)^+ + \big(\tilde{v} - l(M_{s-}, s) + Z_s^n(1)\big)^-\Big] ds \\ &- \int_t^1 \int_{\{-1,1\}} Z_s^n(y) q(ds\, dy), \end{aligned} \tag{3.6.12}$$

where $q(dt\, dy) = d(N_t(y) - 2\lambda t)$. Moreover, the functions g and h automatically satisfy Assumption 3.3.2 and 3.4.4 respectively since both functions come from (4.8) in [20] which satisfies Hypothesis 3.1 of [20].

Compared to (3.5.6), once we determine the value of $Z^n(y)$ for $y \in \{-1, 1\}$, the proof will be done. Since $V^n = Y^n$, we can take the differentiation on V^n to have

$$\begin{aligned} Y_t^n =& -\int_t^1 \int_{\{-1,1\}} \Big[V^n(\tilde{v}, M_{s-}+y, s) - V^n(\tilde{v}, M_{s-}, s)\Big] q(ds\, dy) \\ &- \int_t^1 \Big[V_t^n(\tilde{v}, M_{s-}, s) + \lambda(V^n(\tilde{v}, M_{s-}+1, s) - 2V^n(\tilde{v}, M_{s-}, s) + V_t^n(\tilde{v}, M_{s-}-1, s))\Big] ds. \end{aligned} \tag{3.6.13}$$

Now combining (3.5.5), (3.6.12) and (3.6.13) together, the processes $Z^n(y)$ for $y \in \{-1, 1\}$ can be determined. $\square$

Chapter 4

Trading in limit order market with asymmetric information

4.1 Introduction

In the study of asymmetric information, a continuous time model, Glosten-Milgrom model [25] and Kyle model [32] are influential. In both model, market participants submit market orders to a risk-neutral market maker. Other than the market maker, traders are of two types: informed traders (insiders) and noise traders. The insider possesses the knowledge of value of the asset before the trade and aims to maximise her expected profit by utilizing her private information on the asset. In addition, there are plenty of works extending Glosten-Milgrom model [25] and Kyle model [32] e.g. [6], [5] [18] [33], etc. In these studies, agents are only allowed to submit market orders with unique order size.

In recent years, with growth of electronic exchanges, more than half of the markets use a limit order book (LOB) mechanism to facilitate trades. In LOB market, there is no market maker or specialist who provides bid and ask quotes. There are many papers studying models of LOB. For instance, Roşu [39] considers an equilibrium model that insiders arrive randomly to the market according to an independent Poisson process. Informed investors learn the current value of the asset, and decide whether to buy or sell one unit of the asset, and whether to trade with a market order or a limit order. He illustrates that each informed trader observes the value of mispricing to decide submit which types of orders depending on a given threshold. Moreover, he also states that compared to market orders, limit orders have a smaller price impact by a factor about four. Avellaneda and Stoikov [4] follow early work by Ho and Stoll [29], and represent an inventory management problem that an agent controls the distances between limit orders and mid price to maximise expected terminal profits of the inventory. Guilbaud and Pham [26] propose a framework to study optimal trading strategies in a one-tick pro-rata LOB. The trader decides to submit either market orders or limit orders, which are represented, respectively, by impulse controls and regular controls. Furthermore, several papers, e.g., [16], [27], [41], describe optimal strategies for high frequency trading on cash equities or foreign exchange.

In this chapter, we study an optimal trading problem of a strategic trader, who has a private prediction or signal on asset value. She maybe wrong on her prediction but wants to maximise her trading profits by using this private prediction together with market and limit orders. Rest market participants without this private prediction are aggregated to noisy traders. All of market participants are allowed to submit market or limit orders with multiple sizes. We know that market orders are costly but execution is immediately, and limit orders guarantee the price but execution is uncertain. Hence, the strategic trader faces a trade-off between immediate execution but at a less favourable price, or waiting to be executed but at a better price. From the modelling point of view, for market orders, she tries to control intensities of point processes for associated different order size. On the contrary, we model the strategy of limit orders as continuous controls for order size, due to the fact that these orders can be cancelled immediately with no cost. We also consider the price impact of limit and market orders in our model. In this context, the strategic trader maximises the expected profit over a short time horizon by submitting between limit and market orders.

We formulate the problem as a stochastic control problem and prove that the value function of the strategic trader is a solution to this HJB equation. We also investigate numerically the strategic trader's optimal strategy in a market where limit and market orders have two sizes, small and large. We consider five different scenarios depending on sizes of orders allowed to trade by strategic and noise traders. Our numerical solution shows that the strategic trader will place limit and market buy orders when the magnitude of mispricing, which is the difference between her private prediction on the asset and the current trading price, is higher than a threshold. In certain cases, she may even employ a "round trip" strategy to first submit limit sell orders to push price down, and subsequently uses market buy orders to make profit on low market price. In this round trip of trade, the profits from the market buy is still more than losses from the limit sell.

In this chapter, we first recall market point processes to build up our model and explicitly write down the optimization problem. Then we derive Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem. Finally, we provide a computational algorithm for the resolution of HJB, and illustrate numerically the behavior of the strategic trader under specific scenarios.

4.2 The Model

4.2.1 Marked point process

Consider a measurable space $(E, \mathscr{E})$, and a random sequence $(T_n, \zeta_n)_{n\geq 0} \in [0, \infty) \times E$, where $(T_n)_{n\geq 0}$, starting from $T_0 = 0$, is an increasing sequence of non-anticipating random times to describe the occurrence of events and $\zeta_n \in E$ is a quantity observed at time T_n. We assume that T_n is non-explosive, i.e. $T_n \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$, which guarantees the number of events occurring on any finite time interval, is almost surely finite. The random sequence $(T_n, \zeta_n)_{n\geq 0}$ is called a *marked point process*, where $(T_n)_{n\geq 0}$ is a point process and $(\zeta_n)_{n\geq 0}$ are marks. Define a *counting process* $N_t(K)$ by

$$N_t(K) = \sum_{n\geq 1} \mathbb{I}_{\{T_n \leq t\}} \mathbb{I}_{\{\zeta_n \in K\}}, \quad K \in \mathscr{E}. \tag{4.2.1}$$

We associate to each $K \in \mathscr{E}$ the *counting measure* μ such that

$$\mu((0, t], K) = N_t(K), \quad t \geq 0. \tag{4.2.2}$$

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space where $\mathscr{F}$ satisfies the usual conditions. Denote by $\mathscr{P}$ the $\mathscr{F}$-predictable algebra on $\Omega \times [0, T]$. If any process H is $\mathscr{P} \otimes \mathscr{E}$-measurable satisfying

$$\mathbb{E}\Big[\int_0^T \int_E |H_t(k)|\, \mu(dt, dk)\Big] < \infty, \tag{4.2.3}$$

it follows from [13, Chapter VIII, T14] that there exists a function ϕ_t and an increasing process A with $A_0 = 0$, such that

i) $K \to \phi_t(K)$ is a probability measure on $(E, \mathscr{E})$;

ii) $t \to \phi_t(K)$ is a predictable process;

iii) we have

$$\mathbb{E}\Big[\int_0^T \int_E H_t(k)\mu(dt, dk)\Big] = \mathbb{E}\Big[\int_0^T \int_E H_t(k)\phi_t(dk)dA_t\Big]. \tag{4.2.4}$$

Assumption 4.2.1 The process A is an absolutely continuous increasing process and with respect to time.

The predictable random measure $\phi_t(dk)dA_t$ is denoted by $\nu(dt, dk)$ and called the compensator of μ or dual predictable projection of μ. For H satisfies (4.2.3), we can define the compensated stochastic integral

$$\mathcal{M}_t := \int_0^t \int_E H_r(k)\tilde{\mu}(dr, dk), \tag{4.2.5}$$

where $\tilde{\mu}(dt, dk) := \mu(dt, dk) - \nu(dt, dk)$ is called the *compensated measure.* It follows from [13, Chapter VIII, C4] that $\mathcal{M}$ is a martingale.

4.2.2 Trading model

After recalling marked point processes, let us consider the trading model. In this paper, the microstructure of the market are modelled similar to [6], [18] and Chapter 2. There is a market in continuous time for a risky asset, whose fundamental value denoted by $\tilde{v}$. We assume that $\tilde{v}$ has the upper bound and the lower bound denoted by ν^U and ν^L respectively and $\nu^U > \nu^L$. For simplicity, the risk free interest rate is normalised to 0, i.e. the risk free asset is regarded as the numéraire. There are two types of agents, noisy traders and a strategic trader, all of whom are risk neutral but they have different information. The strategic trader has some private signal about $\tilde{v}$, which is private valuation/prediction of the asset price based on strategic trader's information advantage. This information advantage will lose its value in a future time, say 1. Therefore we assume that the value of $\tilde{v}$ will be revealed to all market participants at time 1. She uses the private value prediction to trade in the market and maximise her expected profit. Rest market participants without this private information are aggregated to noise traders. As the strategic trader always has advance private information on the market, the probability space $(\Omega, \mathbb{P})$ with different filtration accommodates these two types of market participants.

In our model, both market participants are allowed to place market and limit orders. Let us consider the model of market orders for both market participants.

Noisy traders are allowed to place buy or sell market orders with maximum $\bar{m} \in \mathbb{N}$ shares each time. This maximal order size is usually determined by the stock exchange. We assume that the asset is indivisible, therefore the size of buy or sell order k takes values from $K_m = \{1, \dots, \bar{m}\}$ where the subscript m stands for market orders. The arrival of these buy or sell orders are modelled by exogenous Poisson processes. To count the number of buy or sell market orders, we denote by μ^B and μ^S counting measures associated to market buy and sell orders. We also denote by λ^k the buy or sell intensity for order size k. The buy and sell are assumed to be symmetric, i.e. the same λ^k for both buy and sell of order size k. We assume that λ^k is decreasing against k because of large orders arrivals less frequent than small orders. The compensator of $\mu^B(dt, k)$ and $\mu^S(dt, k)$ is $\phi(k)dA_t$, defined in (4.2.4), where $\phi(k) = \frac{\lambda^k}{\sum_{j=1}^m \lambda^j}$ and $dA_t = \sum_{j=1}^m \lambda^j dt$. Here the A represents the arrival of buy/sell market orders of any size and $\phi(k)$ represents the probability that an incoming market buy/sell order is of size k. We denote by Z^B and Z^S market cumulative buy and sell orders respectively. They can be represented as

$$Z^B = \int_0^\cdot \sum_{k=1}^{\bar{m}} k\mu^B(dt, k), \quad Z^S = \int_0^\cdot \sum_{k=1}^{\bar{m}} k\mu^S(dt, k).$$

The cumulative demand is denoted by $Z = Z^B - Z^S$ with initial condition $Z_0 = z \in \mathbb{Z}$.

The other market participant is the *strategic trader*. She has an object to maximise her expected profit out of trading. She applies her private prediction on the asset to place market orders with arbitrary size in K_m. We denote by $\hat{\mu}^B$ and $\hat{\mu}^S$ the counting measures for her market buy and sell orders. For each jump size k, the two processes have intensities $\theta^{B,k}$ and $\theta^{S,k}$ respectively. We assume that these two intensities are bounded by $\bar{\theta} > 0$. The strategic trader controls the values of trading intensities to maximise her expected profit, so the intensities are regarded as control variables. The compensator of $\hat{\mu}^B(dt, k)$ is $\phi(k)dA_t$, where $\phi(k) = \frac{\theta^{B,k}}{\sum_{j=1}^{m}\theta^{B,j}}$ and $dA_t = \sum_{j=1}^{m}\theta^{B,j}dt$. We use similar notation for sell side with superscript S. The cumulative market buy and sell orders, X^B and X^S, for the strategic trader can be represented by

$$X^B = \int_0^{\cdot} \sum_{k=1}^{\bar{m}} k\hat{\mu}^B(dt, k), \quad X^S = \int_0^{\cdot} \sum_{k=1}^{\bar{m}} k\hat{\mu}^S(dt, k). \tag{4.2.6}$$

The cumulative order is denoted by $X = X^B - X^S$ with initial condition $X_0 = 0$.

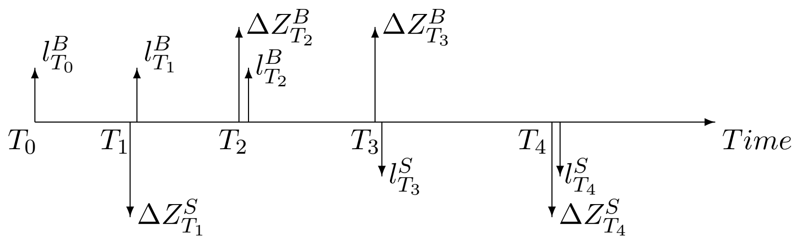
Now let us consider limit orders for both market participants. We assume that limit orders submission or cancellation are free of charge. We also assume that any market order submitted by the strategic trader will be executed against existing orders on the market. Therefore, limit orders from noisy traders are not explicitly modelled.

The limit orders placed by the strategic trader are executed when they are filled by incoming counterpart market orders. After previous execution of limit orders, the strategic trader cancels unexecuted orders and submits new limit orders to wait for next arrival of market orders. We assume that after each arrival of market orders, the strategic trader can submit limit orders, either on buy or sell side, i.e. limit orders are submitted right after the last execution at time T_n, $n \in \mathbb{N}^0$, which is defined as below with initial value $T_0 = 0$. If there is non-execution or partial execution of limit orders, she will cancel the whole or rest orders immediately before placing new limit orders to wait a next execution. Furthermore, for simplicity of the model, we assume that she always submits limit orders at best bid or ask and those have highest priority to be executed compared to other outstanding limit orders. The limit order size is denoted by $l^i \in K_l$, where $K_l = \{0, 1, \ldots, \bar{l}\}$[1] and $\bar{l} \in \mathbb{N}$. Here the subscript l of K_l stands for limit orders. The cumulative submitted buy/sell limit orders up to time $t \in [0, 1)$ is denoted by L^i_t, which is defined as $L^i_t = \sum_{n=0}^{\infty} l^i_{T_n}\mathbb{I}_{\{T_n \le t\}}$, where $i \in \{B, S\}$. In addition, we define $\sigma = \inf\{t > T_n : \Delta Z^S_t \neq 0\}$ and $\tau = \inf\{t > T_n : \Delta Z^B_t \neq 0\}$ to represent the execution time of buy and sell limit orders after last execution time T_n, then the next market order arrives at $T_{n+1} = \sigma \wedge \tau$. As the submission time of limit orders is earlier than the next arrival of market orders, we might need a shifted limit order processes defined as $\tilde{L}^i_t = L^i_{T_n -}$, where $T_n \le t < T_{n+1}$. According to the definition of L^i, $\tilde{L}^i$ jumps at the same time with counterpart X and $\Delta\tilde{L}^i_{T_{n+1}} = \Delta L^i_{T_n} = l^i_{T_n}$ represents how many limit orders are waiting to be executed by incoming market orders at T_{n+1}. Hence, the number of executed limit buy at

[1]If the jump size is 0, it implies the insider does not place any limit order.

time σ can be modelled as $\Delta\tilde{L}^B_\sigma \wedge \Delta Z^S_\sigma$. The executed limit sell can be represented analogously. The aggregated limit orders from the strategic trader can be defined as $\tilde{L} = \tilde{L}^B - \tilde{L}^S$ with initial condition l^B_0 if a limit buy order of size l^B_0 is submitted at time 0 or $-l^S_0$ if a limit sell order of size l^S_0 is submitted at time 0.

Now let us consider an example to understand how the strategic trader places limit orders. There are four market order arrivals from T_1 to T_4. At $T_0 = 0$, she places a limit buy order with $l^B_{T_0}$ size. A market sell order comes at T_1 with $\Delta Z^S_{T_1}$ size. At the same time, if $\Delta Z^S_{T_1} \geq l^B_{T_0}$, the limit buy is fully filled. Otherwise, she cancels unexecuted buy orders and places a new limit buy with size $l^B_{T_1}$. At T_2 a market buy comes in but it cannot fill any limit buy order, so the strategic trader cancels limit buy and places another limit buy order with size $l^B_{T_2}$. After the last market order arrivals at T_4, $L_{T_4} = l^B_{T_0} + l^B_{T_1} + l^B_{T_2} - l^S_{T_3} - l^S_{T_4}$ and $\tilde{L}_t = L_{T_4-} = l^B_{T_0} + l^B_{T_1} + l^B_{T_2} - l^S_{T_3}$ for $t > T_4$.



Now let us consider the aggregation of the informed and noise trades from buy and sell, i.e. $Y^B = Z^B + X^B + \tilde{L}^B$, $Y^S = Z^S + X^S + \tilde{L}^S$ respectively. The total aggregation is $Y = Y^B - Y^S$, which can be regarded as *order flow imbalance* defined in [21], since it is aggregated orders from buy and sell sides.

4.2.3 Pricing rule

We consider market participants who are price takers with respect to an exogenously given pricing rule for shares bought or sold of this stocks within the trading interval. Instead of market participants facing the same price for any order size, they now face a pricing rule that depends on their order demand. Moreover, we also assume that there exists a price impact generated by market and limit orders in our model. The price impact of trades has been extensively studied in the literature. It is usually classified as permanent and temporary price impacts. The permanent price impact is the price change that is due to the information content of the trade. The temporary price impact is the transitory change in prices due to market frictions such as the liquidity effect and the imbalance between demand and supply. In our model, we simply introduce an coefficient of the price impact to combine permanent and temporary price impacts together.

To achieve the properties we assume above, we might define a *pricing rule* which is the function of Y and t such that:

Definition 4.2.2 A function $p : [0, 1] \times \mathbb{Z} \to \mathbb{R}^+$ is a *pricing rule* if

i) $y \to p(t, y)$ is strictly increasing for each $t \in [0, 1)$;

ii) $\lim_{y\to-\infty} p(t, y) = \nu_L$ and $\lim_{y\to+\infty} p(t, y) = \nu_U$ for each $t \in [0, 1]$ where ν_L and ν_U are constants and $\nu_L \leq \nu_U$.

The monotonicity of $y \to p(t, y)$ is natural in asset pricing markets, which implies that when the demand is higher, it generates higher price impact and pushes the price higher. The quantity impact on the price is due to either information effects from the strategic trader or supply/demand imbalances from all market participants.

Now let us consider the price impact on market orders. Çetin et al. [17] study a stochastic supply curve for a security's price as a function of trade size. In Kyle-Back [5], [32] and Glosten-Milgrom [25] models, informed trades cannot be distinguished from non-informed trades, then all trades will generate permanent impact on the price since other agents will believe that a fraction of these trades might contain some private information. For instance, after a buy market order submitted by the strategic trader with ΔX_t^B size, as the price will become $p(t, Y_{t-} + \Delta X_t^B)$ compared to $p(t, Y_{t-})$, the strategic trader needs to pay $\Delta X_t^B p(t, Y_{t-} + \Delta X_t^B)$. For a sell market order, it has a similar result. For temporary price impact on market orders, Alzahrani et al. [2] illustrate that the temporary price impact only contributes one tenth of permanent price impact. On the contrary, Almgren et al. [1] state that permanent and temporary components have equal weight to contribute the total price impact. To compromise different conclusions about permanent/temporary price impacts, we introduce a price impact coefficient $\epsilon^m > 1$, which combine permanent and temporary components together. For instance, after a buy market order submitted by the strategic trader with ΔX_t^B size, the price will become $p(t, Y_{t-} + \epsilon^m \Delta X_t^B)$ compared to $p(t, Y_{t-})$ for any time $t \in [0, 1)$. The permanent price impact is $p(t, Y_{t-} + \Delta X_t^B) - p(t, Y_{t-})$ and the temporary price impact is $p(t, \epsilon^m Y_{t-} + \Delta X_t^B) - p(t, Y_{t-} + \Delta X_t^B)$. The $\epsilon^m - 1$ can be regarded as the temporary price impact factor. If we want to exclude the temporary price impact, we just make ϵ^m be 1. Here, the superscript m of ϵ^m stands for market orders.

Next let us consider the price impact on limit orders. Hautsch and Huang [28] show that limit orders do have significant effects on the price. Cont et al. [21] also state that the price changes are mainly driven by the order flow imbalance, i.e. the aggregation order Y in our model. They also present that there exists a linear relation between price impact and order flow imbalance in short period. To consider the price impact on limit orders, we can also introduce another coefficient $\epsilon^l > 1$, which aggregates permanent and temporary components together. For instance, if the strategic trader submits $\Delta\tilde{L}_t^B$ limit buy orders followed by ΔX_t^S noisy sell market orders, the price will become to $p(t, Y_{t-} + \epsilon^l \Delta\tilde{L}_t^B - \Delta X_t^S)$ compared to price $p(t, Y_{t-})$ for any original time $t \in [0, 1)$.

For a sell limit order, it has a similar result. The $\epsilon^l - 1$ can be regarded as the temporary price impact factor. Here, the superscript l of ϵ^l stands for limit orders.

Before considering the strategic trader's profit, let us first consider an admissible strategy for her.

Definition 4.2.3 The strategy $(X^B, X^S, L^B, L^S; \mathscr{F}^I)$ is admissible, if

i) X^B and X^S are $\mathscr{F}^I$-adapted and integrable marked point processes with initial condition $X_0^B = X_0^S = 0$;

ii) L^B and L^S are $\mathscr{F}^I$-adapted and integrable marked point processes with initial condition which may not be zero;

iii) the $(\mathscr{F}^I, \mathbb{P})$-dual predictable projections of X^B and X^S are absolutely continuous functions of time and intensities bounded by $\bar{\theta} > 0$.

It implies that for each jump size $k \in K_m$ from buy side there exists $\mathscr{F}^I$-intensity $\theta^{B,k}$ such that $X^B - \int_0^\cdot \sum_{k=1}^{\bar{m}} k\theta_r^{B,k} dr = \int_0^\cdot \sum_{k=1}^{\bar{m}} k(\hat{\mu}^B(dr, k) - \theta_r^{B,k} dr)$ is an $\mathscr{F}^I$-martingale. The sell side has a similar result.

4.2.4 Strategic trader's profit

As mentioned earlier, the strategic trader aims to maximise her expected profit. Given an admissible trading strategy from market orders (X^B, X^S), the associated profit from market orders at time 1 is given by

$$\begin{aligned}
&\int_0^1 \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^m \Delta X_r^B)\Big) dX_r^B - \int_0^1 \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^m \Delta X_r^S)\Big) dX_r^S \\
&= \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^m k)\Big) k\hat{\mu}^B(dr, k) - \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^m k)\Big) k\hat{\mu}^S(dr, k).
\end{aligned}$$

Therefore the expected profit of the strategic trader from market orders conditional on her private prediction is

$$\mathbb{E}\Big[\int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^m k)\Big) k\hat{\mu}^B(dr, k) - \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^m k)\Big) k\hat{\mu}^S(dr, k) \Big| \tilde{v}\Big]. \quad (4.2.7)$$

Since pricing rule p is bounded, combined with (4.2.5), we have $\int_0^\cdot \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^m k)\Big) k(\hat{\mu}^B(dr, k) - \theta_r^{B,k} dr)$ is an $\mathscr{F}^I$-martingale. The sell side has a similar result. Therefore, the expected profit from market orders can be expressed as

$$\mathbb{E}\Big[\int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^m k)\Big) k\theta_r^{B,k} dr - \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(u, Y_{r-} - \epsilon^m k)\Big) k\theta_r^{S,k} dr \Big| \tilde{v}\Big]. \quad (4.2.8)$$

According to our assumption, given admissible strategy from limit orders (L^B, L^S), the associated profit from limit orders at time 1 is given by

$$\begin{aligned}
&\int_0^1 \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^l \Delta\tilde{L}_r^B - \Delta Z_r^S)\Big)(d\tilde{L}_r^B \wedge dZ_r^S) \\
&\quad - \int_0^1 \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^l \Delta\tilde{L}_r^S + \Delta Z_r^B)\Big)(d\tilde{L}_r^S \wedge dZ_r^B) \\
&= \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^l \Delta\tilde{L}_r^B - k)\Big)(\Delta\tilde{L}_r^B \wedge k)\mu^S(dr, k) \\
&\quad - \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^l \Delta\tilde{L}_r^S + k)\Big)(\Delta\tilde{L}_r^S \wedge k)\mu^B(dr, k).
\end{aligned}$$

We also know that the expected profit of the strategic trader from limit orders conditional on her information is

$$\begin{aligned}
\mathbb{E}\Big[&\int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^l \Delta\tilde{L}_r^B - k)\Big)(\Delta\tilde{L}_r^B \wedge k)\mu^S(dr, k) \\
&- \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^l \Delta\tilde{L}_r^S + k)\Big)(\Delta\tilde{L}_r^S \wedge k)\mu^B(dr, k)\Big|\tilde{v}\Big].
\end{aligned} \tag{4.2.9}$$

Since pricing rule p is bounded, combined with (4.2.5), we have $\int_0^\cdot \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^l \Delta\tilde{L}_r^B - k)\Big)(\Delta\tilde{L}_r^B \wedge k)(\hat{\mu}^S(dr, k) - \lambda^k dr)$ is an $\mathscr{F}^I$-martingale. The sell side has a similar result. Therefore, the expected limit profit (4.2.9) can be expresses as

$$\begin{aligned}
\mathbb{E}\Big[&\int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} + \epsilon^l \Delta\tilde{L}_r^B - k)\Big)(\Delta\tilde{L}_r^B \wedge k)\lambda^k dr \\
&- \int_0^1 \sum_{k=1}^{\bar{m}} \Big(\tilde{v} - p(r, Y_{r-} - \epsilon^l \Delta\tilde{L}_r^S + k)\Big)(\Delta\tilde{L}_r^S \wedge k)\lambda^k dr\Big|\tilde{v}\Big].
\end{aligned} \tag{4.2.10}$$

4.2.5 Control problem

Let $\mathcal{U}$ be a set of admissible controls, which consists of $[0, \bar{\theta}]^{2\bar{m}} \times [0, \bar{l}]^2$-valued processes. For given

$$u = (\theta^{B,1}, \dots, \theta^{B,\bar{m}}, \theta^{S,1} \dots, \theta^{S,\bar{m}}, l^B, l^S),$$

we define a function for the insider at time t as

$$\begin{aligned}
C(t, y, u) := &\sum_{k=1}^{\bar{m}} \Big\{\big(\tilde{v} - p(t, y + \epsilon^m k)\big)k\theta_t^{B,k} + \big(p(t, y + \epsilon^m k) - \tilde{v}\big)k\theta_t^{S,k} \\
&+ \big(\tilde{v} - p(t, y + \epsilon^l l^B - k)\big)(l^B \wedge k)\lambda^k + \big(p(t, y - \epsilon^l l^S + k) - \tilde{v}\big)(l^S \wedge k)\lambda^k\Big\}.
\end{aligned}$$

Therefore, when the strategic trader uses a control $u \in \mathcal{U}$, the expected profit is

$$J_u(t, y) = \mathbb{E}\Big[\int_t^1 C(r, Y_{r-}, u_r)dr\Big|Y_t = y, \tilde{v}\Big]. \tag{4.2.11}$$

The value function can be defined as

$$V(t,y,\tilde{v}) = \operatorname*{ess\,sup}_{u\in U} J_u(t,y). \tag{4.2.12}$$

Now employing the standard dynamic programming arguments yields the following HJB equation for V:

$$\begin{cases} -V_t(t,y,\tilde{v}) - H(t,y,\tilde{v},V,p) = 0, \\ V(1,y,\tilde{v}) = 0, \end{cases} \tag{4.2.13}$$

where $\tilde{v} \in \{0,1\}$, $(t,y) \in [0,1) \times \mathbb{Z}$. The Hamilton H is defined as

$$H(t,y,\tilde{v},V,p) := H^{(1)}(t,y,\tilde{v},V) + \sup_{u\in U} H^{(2)}(t,y,\tilde{v},V,p), \tag{4.2.14}$$

where $U = [0,\bar{\theta}]^{2\bar{m}} \times [0,\bar{l}]^2$,

$$H^{(1)}(t,y,\tilde{v},V) = \sum_{k=1}^{\bar{m}} \Big[V(t,y+k) - 2V(t,y) + V(t,y-k)\Big]\lambda^k$$

and

$$\begin{aligned} & H^{(2)}(t,y,\tilde{v},V,p) \\ & = \sum_{k=1}^{\bar{m}} \big[V(t,y+k) - V(t,y) + (\tilde{v} - p(t,y+\epsilon^m k))k\big]\theta^{B,k} \\ & + \sum_{k=1}^{\bar{m}} \big[V(t,y-k) - V(t,y) + (p(t,y-\epsilon^m k) - \tilde{v})k\big]\theta^{S,k} \\ & + \sum_{k=1}^{\bar{m}} \big[V(t,y+l^B-k) - V(t,y-k) + \big(\tilde{v} - p(t,y+\epsilon^l l^B - k)\big)(l^B \wedge k)\big]\lambda^k \\ & + \sum_{k=1}^{\bar{m}} \big[V(t,y-l^S+k) - V(t,y+k) + \big(p(t,y-\epsilon^l l^S + k) - \tilde{v}\big)(l^S \wedge k)\big]\lambda^k. \end{aligned}$$

Theorem 4.2.4 *The system (4.2.13) admits a unique bounded solution V continuously differentiable in the time variable. Moreover, there exists a measurable function u^* satisfying*

$$u^*(t,y) = \arg\max_{u\in U} H^{(2)}(t,y,\tilde{v},V,p). \tag{4.2.15}$$

Proof. This proof is motivated by [13, Chapter VII, T3 Theorem]. To simplify notation, we suppress $\tilde{v}$ in V and also p in H throughout the proof. We know U is a compact set. We also notice that the mappings $t \to \theta_t^{B,k}(y)$, $t \to \theta_t^{S,k}(y)$, for any $k \in K_m$ and $(t,\, u) \to C(t,y,u)$ are continuous and bounded. The pricing rule p is bounded as well.

Let l^∞ be the Banach space of real bounded sequences. The supermum norm is defined as $|\mathbf{l}|_{l^\infty} = \sup_{n\in\mathbb{N}} |l_n|$ for $\mathbf{l} = (\ldots, l_{-1},\, l_0,\, l_1, \ldots)$. Then let us consider the following ordinary differential equation in l^∞:

$$\dot{\mathbf{V}}(t) = -\mathbf{H}(t,\mathbf{V}) \;, \quad \mathbf{V}(1) = \mathbf{0},$$

where

$$\begin{aligned}\mathbf{V}(t) &= (\dots, V(t,-1),\, V(t,0),\, V(t,1), \dots),\\ \mathbf{H}(t,\mathbf{V}) &= (\dots,\, H(t,-1,V),\, H(t,0,V),\, H(t,1,V), \dots),\end{aligned} \tag{4.2.16}$$

and H comes from (4.2.14). The symbol $\dot{\mathbf{V}}$ denotes differentiation with respect to t and relative to the sup norm on l^∞. In fact, the differentiability of $t \to \mathbf{V}(t)$ in the l^∞ sense implies the differentiability of $t \to V(t,y)$ for all y in the usual sense, and moreover

$$\dot{\mathbf{V}}(t) = \Big(\dots, \frac{dV(t,-1)}{dt}, \frac{dV(t,0)}{dt}, \frac{dV(t,1)}{dt}, \dots\Big).$$

Now the mappings $\mathbf{V} \to (A^j\mathbf{V})_k$, where $j = \{1,2,3,4\}$ from l^∞ into l^∞ given by

$$\begin{aligned}(A^1\mathbf{V})_k &= V(\cdot, \cdot + k) - V(\cdot,\cdot),\\ (A^2\mathbf{V})_k &= V(\cdot, \cdot - k) - V(\cdot,\cdot),\\ (A^3\mathbf{V})_k &= V(\cdot, \cdot + l^B - k) - V(\cdot,\cdot),\\ (A^4\mathbf{V})_k &= V(\cdot, \cdot - l^S + k) - V(\cdot,\cdot),\end{aligned}$$

are Lipschitz operators for $l^i \in K_L$, $i \in \{B, S\}$, and $k \in K_m$ since $\left|A^j\mathbf{V}\right|_{l^\infty} \le 2\,|\mathbf{V}|_{l^\infty}$. Under the conditions of boundedness for the intensities $\theta^{B,k}$, $\theta^{S,k}$ and λ^k, combined with the profit per unit time C, it is not difficult to show that the mapping from $\mathbb{R}^{4\bar{m}}$ to $\mathbb{R}$ defined by

$$(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4) \to sup_{u\in U}\Big\{\sum_{k=1}^{\bar{m}} x_k^1\theta^{B,k} + \sum_{k=1}^{\bar{m}} x_k^2\theta^{S,k} + \sum_{k=1}^{\bar{m}}(x_k^3 + x_k^4)\lambda^k + C(t,y,u)\Big\} \tag{4.2.17}$$

is Lipschitz and this for all y and $t \in [0,1]$. Therefore, by composition of $(A^j\mathbf{V})_k$ and (4.2.17), i.e. replacing x_k^j by $(A^jV)_k(y)$, $\mathbf{V} \to \mathbf{H}(t,\mathbf{V})$ is Lipschitz map for all $t \in [0,1]$. We can therefore apply the classical results on differential equations on Banach spaces that guarantee the existence of a unique solution of (4.2.13) in l^∞. The last assertion follows from the classical results on measurable selections in [42]. □

Proposition 4.2.5 *We have stated in Theorem 4.2.4 that there exists a function $V(t,y,\tilde{v}) \in C^1$ in time $t \in [0,1]$ satisfying the system* (4.2.13). *Then u_t^* defined by*

$$u_t^* = u^*(t, Y_t), \tag{4.2.18}$$

where u^* comes from (4.2.15)*, is an optimal control.*

Proof. To simplify notation, we ignore $\tilde{v}$ in V. Applying Itô's formula to $V(\cdot, Y_\cdot)$, we obtain

$$V(1,Y_1) = V(t,y) + \int_t^1 V_t(r, Y_{r-})dr + \int_t^1 \sum_{k=1}^{\bar{m}} \big(V(r, Y_{r-} + k) - V(r, Y_{r-})\big)\theta_r^{B,k} dr$$

$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-k)-V(r,Y_{r-})\big)\theta_r^{S,k}dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}+\Delta\tilde L_r^B-k)-V(r,Y_{r-})\big)\lambda^k dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-\Delta\tilde L_r^S+k)-V(r,Y_{r-})\big)\lambda^k dr+M_1-M_t,$$

where

$$M=\int_0^\cdot \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}+k)-V(r,Y_{r-})\big)(\hat\mu^B(dr,k)-\theta_r^{B,k}dr)$$
$$+\int_0^\cdot \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-k)-V(r,Y_{r-})\big)(\hat\mu^S(dr,k)-\theta_r^{S,k}dr)$$
$$+\int_0^\cdot \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}+\Delta\tilde L_r^B-k)-V(r,Y_{r-})\big)(\mu^S(dr,k)-\lambda^k dr)$$
$$+\int_0^\cdot \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-\Delta\tilde L_r^S+k)-V(r,Y_{r-})\big)(\mu^B(dr,k)-\lambda^k dr)$$

is a martingale. Now, we add $\int_t^1 C(t,Y_{r-},u_r)dr$ on both sides in above equation, we have

$$\int_t^1 C(r,Y_{r-},u_r)dr=-V(1,Y_1)+V(t,y)+\int_t^1 V_t(r,Y_{r-})dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}+k)-V(r,Y_{r-})\big)\theta_r^{B,k}dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-k)-V(r,Y_{r-})\big)\theta_r^{S,k}dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}+\Delta\tilde L_r^B-k)-V(r,Y_{r-})\big)\lambda^k dr$$
$$+\int_t^1 \sum_{k=1}^{\bar m} \big(V(r,Y_{r-}-\Delta\tilde L_r^S+k)-V(r,Y_{r-})\big)\lambda^k dr$$
$$+\int_t^1 C(t,Y_{r-},u_r)dr+M_1-M_t.$$

Moreover, taking the conditional expectation with respect to $\mathscr{F}_t^I$ on both sides, by the HJB equation (4.2.13), we have

$$J_u(t,y)\le V(t,y)-V(1,Y_1)=V(t,y), \tag{4.2.19}$$

where the equality is attained at u^* by the definition in (4.2.18), and the identity is because of the terminal condition of V. □

4.3 Numerical example

In this section, we solve (4.2.13) numerically then illustrate the strategic trader's optimal value and strategy. We assume that the asset price $\tilde v$ follows a Bernoulli distribution with $\mathbb{P}(\tilde v=0)=p$ and

$\mathbb{P}(\tilde{v} = 1) = 1 - p$. To ease of notation, we suppress $\tilde{v}$ in each function below.

4.3.1 Pricing function

In our model, the pricing function defined in Definition 4.2.2 is general. To observe the behaviour of the strategic trader, we need to specify a pricing function in the numerical scheme. We can borrow the pricing rule from literature [18], who illustrates that when there only exists market orders with unique order size, the pricing function is $p(t, Z_t) = \mathbb{E}\Big[\tilde{v}|\mathscr{F}_t\Big]$ set by a market maker to achieve an equilibrium. Even though we do not consider the equilibrium here, we want to see the impact of limit orders with the same order size as well on strategic trader's optimal value once limit orders are allowed. In particular, $p(t, Z_t) = \mathbb{E}\Big[\tilde{v}|\mathscr{F}_t\Big]$ describes market's implied probability that $\tilde{v} = 1$ since $\tilde{v}$ has Bernoulli distribution. The price $p(t, Z_t)$ is market's evaluation of the asset at time t and satisfies (3.5) in [18] due to the Markov property in Z. Here we assume that $p(t, Y_t)$ is market's evaluation and satisfies

$$\begin{cases} p_t + \sum_{k=1}^{\bar{m}} \Big\{ p(t, y+k) - p(t, y) + p(t, y-k) \Big\} \lambda^k = 0, \quad (t, y) \in [0, 1) \times \mathbb{Z}, \\ p(1, y) = P(y), \quad y \in \mathbb{Z}, \end{cases} \tag{4.3.1}$$

where

$$P(y) := \begin{cases} 0 & y < z \\ 1 & y \geq z \end{cases}, \tag{4.3.2}$$

for any value of $z \in \mathbb{Z}$.

4.3.2 Numerical results

Let us introduce parameters in the computational scheme which numerically solve the system (4.2.13). The time interval $[0, 1]$ can be discretised with time step $\Delta t = 1/N$ and a regular time grid $\mathbb{T}_N = \{t_n = n\Delta t,\ n = 0, \ldots, N\}$. We assume that the interval of aggregated demand Y is $[-M_Y, M_Y]$ which implies the state space Y is truncated at $-M_Y$ and M_Y for large M_Y. Then we can discretise the state space with size step $\Delta y = 2M_Y/N_Y$ and a finite regular grid $\mathbb{Y}_{M_Y} = \{y_m = -M_Y + m\Delta y,\ m = 0, \ldots, N_Y\}$. To find the optimal strategy for the strategic trader, we need to perform the algorithm (4.4.4) with parameters shown in Table 4.1. The details of the numerical scheme are deferred to the Appendix.

In Table 4.1, we assume that $\tilde{v}$ follows Bernoulli distribution with 0 or 1 values. When the value $p = 0.5$, it illustrates that $\tilde{v}$ is an unbiased Bernoulli random variable. In addition, we assume that the strategic trader predicts $\tilde{v}$ being equal to 1 by using her advanced information. As the whole trading period is $[0, 1]$, we set terminal time T as 1. For simplicity reason, we also suppose that all of agents are allowed to place orders with maximum two units of orders. One unit size can be regarded as small orders, which arrives frequently with intensity $\lambda^1 = 200$. Two units size can be regarded as large orders with intensity $\lambda^2 = 20$, which only has one tenth of λ^1 as large orders arrivals much less frequent than small orders. The intensities of market orders $\theta^{i,k}$, where $i \in \{B, S\}$ and $k \in \{1, 2\}$, is bounded by $\bar{\theta}$, which is 400. We assume that the number of time partitions is $N = 5,000$, which means the time interval $\Delta t = 1/N = 0.0002$. In addition, we assume that the interval of aggregated orders is defined as $1/\sqrt{2\lambda^1}$, which is motivated by [6] and [18]. The value of M_Y is three times of standard deviation of results obtained by 10^6 times of Monte Carlo simulation for the difference of two independent of Poisson processes with the intensity $\lambda^1 + \lambda^2$. Once we have values of Δy and M_Y, the value of N_Y will be determined. Since $p = 0.5$ and the difference of independent Poisson processes, the value of z in (4.3.2) is the middle point of $[-M_Y, M_Y]$ which is 0. Finally, we assume that the price impact of market orders will generated three times effect compared to limit ones, i.e. 6 and 2 respectively.

Parameter	Value	Parameter	Value	Parameter	Value
p	0.5	T	1	z	0
$\tilde{v}$	1	λ^1	200	λ^2	20
$\bar{\theta}$	400	N	5000	Δt	0.0002
Δy	$\frac{1}{\sqrt{2\lambda^1}} = 0.05$	M_Y	72	N_Y	2880
ϵ^m	6	ϵ^l	2		

Table 4.1: Parameters for numerical scheme

Here we consider five different scenarios listed in Table 4.2. The range of scenarios is from all of market participants only allowed to place market orders with small unit order size to all of them allowed to place market and limit orders with small and large order size. The second column is the values of $V^{\Delta t, \Delta y. M_y}(0, 0)$ defined in (4.4.4) when the strategic trader applies her optimal strategy to trade each time. The first three scenarios are classified to the first group and the rest are aggregated to the second group as the values in last two scenarios have a significant jump. The reason will be explained later. In the each of group, the values gradually increase along with extending trading options. Now let us consider these five scenarios one by one.

The numerical results are listed in section 4.4.3, the y-axis is the trading period from 0 to 1 and the x-axis is the number of order units for aggregated orders. In general, as the private prediction

Scenarios	Value at $y = 0$ and $t = 0$
1 Market Order Size Only	0.2658
1 Market Order Size, 1 Limit Order Size	0.3508
2 Market Order Size, 1 Limit Order Size	0.4024
1 Market Order Size, 2 Limit Order Size	8.2242
2 Market Order Size, 2 Limit Order Size	8.8654

Table 4.2: Optimal values for different scenarios

of $\tilde{v}$ is 1, the strategic trader always wants to push the aggregated demand y above the $z = 0$ due to (4.3.2) and the definition of Z.

In the Figure 4.3, we display the optimal strategy when only both market orders with small order size are allowed. There are two regions for the strategy of market buy, active region and inactive region. Since the control problem is of bang-bang type, optimal intensity is either maximal $\bar{\theta}$ in the active region or 0 in the inactive region. At the beginning of the trade, she is patient to place market buy orders. However, as time passed by, she becomes more and more impatient and tries to place market orders once the number of aggregated orders is less than zero, i.e. the number of sell more than the number of buy. In market sell side, the strategic trader does not place any orders to sell since it incurs a permanent loss, which is not the optimal strategy for her.

In the Figure 4.4, we display the optimal strategy when market and limit orders in small order size are allowed. The strategy in limit buy orders is very similar as market buy orders. At the same time, she does not place any market and limit sell orders. The reason of no limit sell orders submitted can be demonstrated by an example as below. For instance, at time t, when the strategic trader places the limit sell with small order size $l^s = 1$ which is fully filled by the following market buy with order size $k = 1$ submitted by noisy traders, the strategic trader suffers the loss when $\epsilon^m = 6$ and $\epsilon^l = 2$ such that

$$\text{Loss} = (p(t, y - \epsilon^l l^s + k) - \tilde{v})(l^s \wedge k) = p(t, y - 1) - \tilde{v}.$$

Since the price of the asset is less than $\tilde{v} = 1$ for any y and t, the value of the above identity is non-positive. After posting a limit sell followed by a market buy order with the same size, the aggregated order is unchanged, i.e. $y - 1 + 1 = y$, due to the negative one contributed from the limit sell and the positive one contributed from the market buy. Next the strategic trader places a market buy order with small order size $k = 1$ at time $t' > t$ to obtain the profit such that

$$\text{Profit} = (\tilde{v} - p(t, y + \epsilon^m k))k = \tilde{v} - p(t, y + 6).$$

For simplicity, we assume that there is no any other orders coming in between the time interval t and t'. The net profit is

$$\begin{aligned}\text{Net Profit} &= \text{Profit} + \text{Loss} \\ &= p(t, y-1) - p(t, y+6) < 0.\end{aligned}$$

The negative net profit also happens when she submits limit sell orders followed by limit buy orders. Hence, submitting limit sell orders is not the optimal strategy as the profit from the market buy is less than the loss from the limit sell in this scenario. Furthermore, comparing to the strategy in Figure 4.3, we find that the strategic trader is more patient at early of the trade as she has one more option, limit buy orders, to make profits.

In Figure 4.5, we display the optimal strategy when market orders with both order sizes and limit orders with only small order size are allowed. The pattern of the trading strategy is similar as the previous two scenarios. The main difference is that the strategic trader is the most patient among these three scenarios as she has the most trading options in the third scenarios.

Now let us consider last two scenarios together. In Figure 4.6 and 4.7, the trading strategies in market buy, market sell and limit buy orders are similar to three previous scenarios. However, the values of strategic trader's profit in the fourth and fifth scenarios, 8.2242 and 8.8654 respectively, are much larger than values in the three previous scenarios. The main reason is the strategy in limit sell orders. It is easy to use an example to explain the reason. For instance, at time t, when the strategic trader places a limit sell order with large order size $l^s = 2$ which is partially filled by the following a market buy order with size $k = 1$ submitted by noisy traders, the strategic trader suffers a loss such that

$$\text{Loss} = (p(t, y - \epsilon^l l^s + k) - \tilde{v})(l^s \wedge k) = p(t, y-3) - \tilde{v},$$

which is non-positive. Now the number of the aggregated order is $y - 2 + 1 = y - 1$ due to negative two contributed from the limit sell side and positive one contributed from the market buy side. Next the strategic trader places a market buy order with large size $k = 2$ at time $t' > t$ to obtain a profit such that

$$\text{Profit} = (\tilde{v} - p(t, y - 1 + \epsilon^m k))k = 2(\tilde{v} - p(t, y+11)).$$

We also assume that there is no any other orders coming in between the time interval t and t'. Hence the net profit is

$$\begin{aligned}\text{Net Profit} &= \text{Profit} + \text{Loss} \\ &= 2(\tilde{v} - p(t', y+11)) + p(t, y-3) - \tilde{v}\end{aligned}$$

$$= \tilde{v} - p(t', y+11) + p(t, y-3) - p(t', y+11).$$

As the pricing rule is non-decreasing against y and is always less or equal to $\tilde{v}$, the value of the net profit might be positive when the value of $\tilde{v} - p(t', y+11)$ is larger than the value of $p(t', y+11) - p(t, y-3)$. Therefore, the strategic trader will place market orders with large size when it happens, which is demonstrated in the fourth plot in the Figure 4.6 and 4.7. In summary, the strategic trader submits limit sell orders with large size to push the trading price down and suffers a loss, and places market buy orders with large order size to take advantage of low trading price and make immediate profit which is larger than the previous loss. Therefore, comparing to do nothing in the limit sell, the behaviour of "round trip" trading is an optimal strategy for her.

4.4 Appendix

4.4.1 Numerical scheme

In this section, we will consider the details of the numerical scheme to solve the system (4.2.13). Besides introducing parameters listed in Table 4.1, we defined a truncated function denoted by $\varphi(y) := -M_Y \vee (y \wedge M_Y)$. Now let us define an operator associated to the (4.3.1): given a [0,1]-real valued function ϕ on $[0,1] \times \mathbb{R}$, we define

$$\mathcal{P}^{\Delta t, \Delta y, M_Y}(t, y, \phi) := \phi(t, y) + \Delta t \sum_{k=1}^{\bar{m}} \Big\{ \phi\big(t, \varphi(y + k\Delta y)\big) - \phi(t, y) + \phi\big(t, \varphi(y_m - k\Delta y)\big) \Big\} \lambda^k.$$

We also define an operator associated to the (4.2.13): given a real valued function ψ on $[0,1] \times \mathbb{R}$, we define

$$\mathcal{S}^{\Delta t, \Delta y, M_Y}(t, y, \psi, \phi) := \psi(t, y) + \Delta t \times \hat{H}(t, y, \psi, \phi),$$

where

$$\hat{H}(t, y, \psi, \phi) := \hat{H}^{(1)}(t, y, \psi) + \sup_{u \in U} \hat{H}^{(2)}(t, y, \psi, \phi)$$

such that

$$\hat{H}^{(1)}(t, y, \psi) := \sum_{k=1}^{\bar{m}} \Big[\psi\big(t, \varphi(y + k\Delta y)\big) - 2\psi(t, y) + \psi\big(t, \varphi(y - k\Delta y)\big) \Big] \lambda^k, \tag{4.4.1}$$

and

$$\begin{aligned}\hat{H}^{(2)}(t, y, \psi, \phi) :=& \sum_{k=1}^{\bar{m}} \Big[\psi\big(t, \varphi(y + k\Delta y)\big) - \psi(t, y) + \big(\tilde{v} - \phi(t, \varphi(y + \epsilon^m k\Delta y))\big) k\Delta y \Big] \theta^{B,k} \\ &+ \sum_{k=1}^{\bar{m}} \Big[\psi\big(t, \varphi(y - k\Delta y)\big) - \psi(t, y) + \big(\phi(t, \varphi(y - \epsilon^m k\Delta y)) - \tilde{v}\big) k\Delta y \Big] \theta^{S,k}\end{aligned}$$

$$+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y+(l^B-k)\Delta y)\big)-\psi\big(t,\varphi(y-k\Delta y)\big)$$
$$+\big(\tilde{v}-\phi(t,\varphi(y+(\epsilon^l l^B-k)\Delta y))\big)(l^B\wedge k)\Delta y\Big]\lambda^k$$
$$+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y-(l^S-k)\Delta y)\big)-\psi\big(t,\varphi(y+k\Delta y)\big)$$
$$+\big(\phi\big(t,\varphi(y-(\epsilon^l l^S-k)\Delta y)\big)-\tilde{v}\big)(l^S\wedge k)\Delta y\Big]\lambda^k. \tag{4.4.2}$$

Now we can approximate the solution p in (4.3.1) by the function $p^{\Delta t,\Delta y,M_Y}$ on $(t_n,y_m)\in\mathbb{T}_N\times\mathbb{Y}_M$ solution to the computational scheme

$$\begin{cases} p^{\Delta t,\Delta y,M_Y}(t_n,y_m)=\mathcal{P}^{\Delta t,\Delta y,M_Y}(t_{n+1},y_m,p^{\Delta t,\Delta y,M_Y}) \\ p^{\Delta t,\Delta y,M_Y}(t_N,y_m)=P(y_m). \end{cases} \tag{4.4.3}$$

We can also approximate the solution V in (4.2.13) by the function $V^{h,\Delta y,M_Y}$ on $(t_n,y_m)\in\mathbb{T}_N\times\mathbb{Y}_M$ solution to the computational scheme

$$\begin{cases} V^{\Delta t,\Delta y,M_Y}(t_n,y_m)=\mathcal{S}^{\Delta t,\Delta y,M_Y}(t_{n+1},y_m,V^{\Delta t,\Delta y,M_Y},p^{\Delta t,\Delta y,M_Y}), \\ V^{\Delta t,\Delta y,M_Y}(t_N,y_m)=0. \end{cases} \tag{4.4.4}$$

The algorithm can be described explicitly in backward induction by the following pseudo-code: for each $y_m\in\mathbb{Y}_{M_Y}$,

- for $t_N=1$, set $p^{\Delta t,\Delta y,M_Y}(t_N,y_m)=P(y_m)$;
- for $n=N-1,\dots,0$, assign $p^{\Delta t,\Delta y,M_Y}(t_n,y_m)$ by computing $\mathcal{P}^{\Delta t,\Delta y,M_Y}(t_{n+1},y_m,p^{\Delta t,\Delta y,M_Y})$;
- for $t_N=1$, set $V^{\Delta t,\Delta y,M_Y}(t_N,y_m)=0$;
- for $n=N-1,\dots,0$, compute

  $$\hat{H}^{(1)}(t_{n+1},y_m,V^{\Delta t,\Delta y,M_Y}) \text{ and } \hat{H}^{(2)}(t_{n+1},y_m,V^{\Delta t,\Delta y,M_Y},p^{\Delta t,\Delta y,M_Y}),$$

  and store $\theta^{B,k,\star}$, $\theta^{S,k,\star}$, $l^{B,\star}$ and $l^{S,\star}$ the argmax. Finally assign $V^{\Delta t,\Delta y,M_Y}(t_n,y_m)$ by computing

  $$V^{\Delta t,\Delta y,M_Y}(t_{n+1},y_m)+\Delta t\Big(\hat{H}^{(1)}(t_{n+1},y_m,V^{\Delta t,\Delta y,M_Y})+\hat{H}^{(2)}(t_{n+1},y_m,V^{\Delta t,\Delta y,M_Y},p^{\Delta t,\Delta y,M_Y})\Big).$$

The convergence of the numerical scheme (4.4.4) is by showing the monotonicity, stability, and consistency properties of this scheme. The proof is provided below. Now we can confirm that the solution $V^{\Delta t,\Delta y,M_Y}$ to the numerical scheme (4.4.4) converges locally uniformly to V on $[0,1)\times\mathbb{Z}$, as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$.

4.4.2 Proof of convergence

In this section, the proof follows [26, Section 4.2] with proper extension. To ease of notation, we suppress the parameter $\tilde{v}$ for each function in this section. We denote by C_b^1 the set of bounded continuously differentiable functions on $[0,1]\times\mathbb{R}$ with bounded derivatives, and $C_{[0,1]}^1$ the set of [0,1]-bounded continuously increasing functions on $[0,1]\times\mathbb{R}$ with bounded derivatives.

Assumption 4.4.1 To prove the convergence of $V^{\Delta t,\Delta y,M_Y}$ to V as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$, we assume that

i) the functions $\phi\in C_{[0,1]}^1$ and $\psi\in C_b^1$;

ii) the solution $p^{\Delta t,\Delta y,M_Y}$ to the numerical scheme (4.4.3) converges locally uniformly to p for (4.3.1) on $[0,1)\times\mathbb{Z}$ as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$.

Remark 4.4.2 Here we assume that item ii) in Assumption 4.4.1 holds without a detailed proof as the way to finish the proof is similar as the technique in the proof of the convergence from $V^{\Delta t,\Delta y,M_Y}$ to V illustrated below. It is needed to show monotonicity, stability and consistency properties of the scheme (4.4.3). Combining these three properties, we can confirm that there exists a convergence from (4.4.3) to (4.3.1) as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$.

Now let us study the convergence of numerical scheme (4.4.4) by showing monotonicity, stability and consistency properties of this scheme.

Lemma 4.4.3 *(Monotonicity). For any $\Delta t>0$ s.t. $\Delta t<\left(2\sum_{k=1}^m(\lambda^k+\bar{\theta})\right)^{-1}$, the operator $\mathcal{S}^{\Delta t,\Delta y,M_Y}(t,y,\psi,\phi)$ is non-decreasing in ψ, i.e. for any $\phi\in C_{[0,1]}^1$, ψ_1 and $\psi_2\in C_b^1$, s.t. $\psi_1\le\psi_2$:*

$$\mathcal{S}^{\Delta t,\Delta y,M_Y}(t,y,\psi_1,\phi)\le\mathcal{S}^{\Delta t,\Delta y,M_Y}(t,y,\psi_2,\phi),\quad (t,y)\in[0,1)\times\mathbb{R}.$$

Proof. We see that the $\mathcal{S}^{h,\Delta y,M_Y}(t,y,\psi)$ can be written as

$$\begin{aligned}
&\mathcal{S}^{\Delta t,\Delta y,M_Y}(t,y,\psi,\phi)\\
=&\psi(t,y)+\Delta t\times\Big(\hat{H}^{(1)}(t,y,\psi)+\sup_{u\in U}\hat{H}^{(2)}(t,y,\psi,\phi)\Big)\\
=&\psi(t,y)+\sup_{u\in U}\Bigg\{\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y+k\Delta y)\big)-\psi(t,y)+\big(\tilde{v}-\phi(t,\varphi(y+\epsilon^m k\Delta y))\big)k\Delta y\Big]\Delta t\theta^{B,k}\\
&+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y-k\Delta y)\big)-\psi(t,y)+\big(\phi(t,\varphi(y-\epsilon^m k\Delta y))-\tilde{v}\big)k\Delta y\Big]\Delta t\theta^{S,k}\\
&+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y+(l^B-k)\Delta y)\big)-\psi\big(t,y\big)+\big(\tilde{v}-\phi(t,\varphi(y+(\epsilon^l l^B-k)\Delta y))\big)(l^B\wedge k)\Delta y\Big]\Delta t\lambda^k\\
&+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y-(l^S-k)\Delta y)\big)-\psi\big(t,y\big)+\big(\phi\big(t,\varphi(y-(\epsilon^l l^S-k)\Delta y)\big)-\tilde{v}\big)(l^S\wedge k)\Delta y\Big]\Delta t\lambda^k\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
=\sup_{u\in U}\Bigg\{&\psi(t,y)+\sum_{k=1}^{\bar{m}}\Bigg\{-(\theta^{B,k}+\theta^{S,k}+2\lambda^k)\Delta t\psi(t,y)\\
&+\Big[\psi\big(t,\varphi(y+k\Delta y)\big)+\big(\tilde{v}-\phi(t,\varphi(y+\epsilon^m k\Delta y))\big)k\Delta y\Big]\Delta t\theta^{B,k}\\
&+\Big[\psi\big(t,\varphi(y-k\Delta y)\big)+\big(\phi(t,\varphi(y+\epsilon^m k\Delta y))-\tilde{v}\big)k\Delta y\Big]\Delta t\theta^{S,k}\Bigg\}\\
&+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y+(l^B-k)\Delta y)\big)+\big(\tilde{v}-\phi(t,\varphi(y+(\epsilon^l l^B-k)\Delta y))\big)(l^B\wedge k)\Delta y\Big]\Delta t\lambda^k\\
&+\sum_{k=1}^{\bar{m}}\Big[\psi\big(t,\varphi(y-(l^S-k)\Delta y)\big)+\big(\phi(t,\varphi(y-(\epsilon^l l^S-k)\Delta y)-\tilde{v})\big)(l^S\wedge k)\Delta y\Big]\Delta t\lambda^k\Bigg\}.
\end{aligned}
$$

From the expression above, it is clear that $\mathcal{S}^{\Delta t,\Delta y,M_Y}(t,y,\psi,\phi)$ in ψ is monotone once $\Delta t<\Big(2\sum_{k=1}^{m}(\lambda^k+\bar{\theta})\Big)^{-1}$. □

Lemma 4.4.4 *(Stability) For any $\Delta t,\Delta y,M_Y>0$ there exists a unique solution $V^{\Delta t,\Delta y,M_Y}$ to the numerical scheme* (4.4.4)*, and the sequence $(V^{\Delta t,\Delta y,M_Y})$ is uniformly bounded for any $(t_n,y_m)\in\mathbb{T}_N\times\mathbb{Y}_M$.*

Proof. Existence and uniqueness of $V^{\Delta t,\Delta y,M_Y}$ follows from the backward scheme (4.4.4). Let us now consider the uniform bounds. Since $\theta^{i,k}$, l^i, λ^k and $p^{\Delta t,\Delta y,M_Y}$ for $i\in\{B,S\}$ are bounded, there always exists a constant γ to make $V^{\Delta t,\Delta y,M_Y}<\gamma$. □

Lemma 4.4.5 *(Consistency) For all $(t,y)\in[0,1)\times\mathbb{Z}$, we have $\psi\in C_b^1$, $\phi\in C_{[0,1]}^1$ and*

$$\lim_{\substack{(\Delta t,\Delta y,M_Y)\to(0,1,\infty)\\(t',y')\to(t,y)}}\phi\to\hat{\phi},$$

where $\hat{\phi}$ is the function on $[0,1]\times\mathbb{Z}$, then we can show

$$\lim_{\substack{(\Delta t,\Delta y,M_Y)\to(0,1,\infty)\\(t',y')\to(t,y)}}\frac{1}{\Delta t}\Big[\psi(t',y')-\mathcal{S}^{\Delta t,\Delta y,M_Y}(t'+\Delta t,y',\psi,\phi)\Big]=-\psi_t(t,y)-H(t,y,\psi,\hat{\phi}).\tag{4.4.5}$$

Proof. We have all $(t',y')\in[0,1)\times\mathbb{R}$,

$$\frac{1}{\Delta t}\Big[\psi(t',y')-\mathcal{S}^{\Delta t,\Delta y,M_Y}(t'+\Delta t,y',\psi,\phi)\Big]=\frac{1}{\Delta t}\Big[\psi(t',y')-\psi(t'+\Delta t,y')\Big]-\hat{H}(t'+\Delta t,y',\psi,\phi).$$

The first term converges trivially to $-\psi_t(t,y)$ as Δt goes to 0 and (t',y') goes to (t,y). To complete this proof, we just need to show the convergence of $\hat{H}(t'+\Delta t,y',\psi,\phi)$ to $H(t,y,\psi,\hat{\phi})$ as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$ and (t',y') goes to (t,y). Alternatively, we need to prove the convergence of $\hat{H}^{(1)}(t'+\Delta t,y',\psi)$ and $\hat{H}^{(2)}(t'+\Delta t,y',\psi,\phi)$ to $H^{(1)}(t,y,\psi)$ and $H^{(2)}(t,y,\psi,\hat{\phi})$ respectively.

Now let us consider the convergence of $\hat{H}^{(1)}(t'+\Delta t,y',\psi)$ to $H^{(1)}(t,y,\psi)$. The convergence of the first term in $\hat{H}^{(1)}(t'+\Delta t,y',\psi)$ to the corresponding term in $H^{(1)}(t,y,\psi)$ is such that for three bounded constants η_1, η_2 and η_3,

$$\Bigg|\sum_{k=1}^{\bar{m}}\lambda^k\Big[\psi\big(t'+\Delta t,\varphi(y'+k\Delta y)\big)-\psi(t,y+k)\Big]\Bigg|\le\sum_{k=1}^{\bar{m}}\lambda^k\Big|\psi\big(t'+\Delta t,\varphi(y'+k\Delta y)\big)-\psi(t,y+k)\Big|$$

$$
\begin{aligned}
&= \sum_{k=1}^{\bar{m}} \lambda^k \Big| \psi(t' + \Delta t, y' + k\Delta y) - \psi(t, y+k) \Big| \mathbb{I}_{\{y'+k\Delta y \in [-M_Y, M_Y]\}} \\
&\quad + \sum_{k=1}^{\bar{m}} \lambda^k \Big| \psi(t' + \Delta t, M_Y) - \psi(t, y+k) \Big| \mathbb{I}_{\{y'+k\Delta y > M_Y\}} \\
&\quad + \sum_{k=1}^{\bar{m}} \lambda^k \Big| \psi(t' + \Delta t, -M_Y) - \psi(t, y+k) \Big| \mathbb{I}_{\{y'+k\Delta y < -M_Y\}} \\
&\le \eta_1 \left| \psi^{(1)} \right|_\infty \left(\Delta t + \left| y + 1 - y' - \Delta y \right| \right) + \eta_2 \left| \psi \right|_\infty \mathbb{I}_{\{y'+\epsilon^m k\Delta y \ge M_Y\}} + \eta_3 \left| \psi \right|_\infty \mathbb{I}_{\{y'+\epsilon^m k\Delta y \le -M_Y\}},
\end{aligned}
$$

where ψ is bounded by $|\psi|_\infty$ and the derivative of ψ is bounded by $\left|\psi^{(1)}\right|_\infty$ because of Assumption 4.4.1. Once $(\Delta t, \Delta y, M_Y) \to (0, 1, \infty)$ and $(t', y') \to (t, y)$, the convergence is proved. Now let us consider the second term in $\hat{H}^{(1)}(t' + \Delta t, y', \psi)$ such that for a bounded constant η_4,

$$
\left| \sum_{k=1}^{\bar{m}} \Big\{ \psi(t' + \Delta t, y') - \psi\big(t, y\big) \Big\} \right| \le \sum_{k=1}^{\bar{m}} \Big| \psi(t' + \Delta t, y') - \psi\big(t, y\big) \Big| \le \eta_4 \left| \psi^{(1)} \right|_\infty \left(\Delta t + \left| y - y' \right| \right),
$$

which converges to 0 as $(\Delta t, \Delta y, M_Y) \to (0, 1, \infty)$ and $(t', y') \to (t, y)$. For the third term, it can be done as similar as the first term. Hence, combining these three results, we can confirm

$$
\lim_{\substack{(\Delta t, \Delta y, M_Y) \to (0,1,\infty) \\ (t', y') \to (t, y)}} \hat{H}^{(1)}(t' + \Delta t, y', \psi) = H^{(1)}(t, y, \psi).
$$

Next let us consider the convergence from $\sup_{u \in U} \hat{H}^{(2)}(t' + \Delta t, y', \psi, \phi)$ to $\sup_{u \in U} H^{(2)}(t, y, \psi, \hat{\phi})$. The convergence of the terms of market buy in $\sup_{u \in U} \hat{H}^{(2)}(t' + \Delta t, y', \psi)$ to the corresponding term in $\sup_{u \in U} H^{(2)}(t, y, \psi)$ is such that

$$
\begin{aligned}
&\Bigg| \sum_{k=1}^{\bar{m}} \sup_{\theta^{B,k} \in [0, \bar{\theta}]} \bigg\{ \Big[\psi\big(t' + \Delta t, \varphi(y' + k\Delta y)\big) - \psi(t' + \Delta t, y') + \big(\tilde{v} - \phi(t' + \Delta t, \varphi(y' + \epsilon^m k \Delta y))\big) k \Delta y \Big] \theta^{B,k} \bigg\} \\
&\quad - \sum_{k=1}^{\bar{m}} \sup_{\theta^{B,k} \in [0, \bar{\theta}]} \bigg\{ \Big[\psi(t, y+k) - \psi(t, y) + (\tilde{v} - \hat{\phi}(t, y + \epsilon^m k)) k \Big] \theta^{B,k} \bigg\} \Bigg| \\
&\le \sum_{k=1}^{\bar{m}} \theta^{B,k} \Big| \psi\big(t' + \Delta t, \varphi(y' + k\Delta y)\big) - \psi(t, y+k) \Big| + \sum_{k=1}^{\bar{m}} \theta^{B,k} \Big| \psi(t' + \Delta t, y') - \psi(t, y) \Big| \\
&\quad + \sum_{k=1}^{\bar{m}} k \theta^{B,k} \Big| \phi\big(t' + \Delta t, \varphi(y' + \epsilon^m k \Delta y)\big) \Delta y - \hat{\phi}(t, y + \epsilon^m k) \Big|.
\end{aligned}
$$

As the control problem is of bang-bang type, the optimal intensity is $\bar{\theta}$ when the coefficients are positive. Hence applying the inequality $\left| x_+ - x'_+ \right| \le |x - x'|$, we can obtain the above inequality. As the convergence of the first and second terms on the right hand side of the above inequality have been proved in last paragraph, and the convergence of the third term automatically holds due to Assumption 4.4.1, we can confirm that the terms of market buy in $\hat{H}^{(2)}(t' + \Delta t, y', \psi, \phi)$ converges to the corresponding terms in $H^{(2)}(t, y, \psi, \hat{\phi})$. We can apply the same technique to finish the rest of proofs.

□

Proposition 4.4.6 *(Convergence) The solution $V^{\Delta t,\Delta y,M_Y}$ to the numerical scheme* (4.4.4) *converges locally uniformly to V on $[0,1)\times\mathbb{Z}$ as $(\Delta t,\Delta y,M_Y)$ goes to $(0,1,\infty)$.*

Proof. Given the above monotonicity, stability and consistency properties, the convergence of the sequence $(V^{\Delta t,\Delta y,M_Y})$ towards V, which is the unique bounded viscosity solution to (4.2.13), follows from [9]. We claim that V^* and V_* defined as

$$V^*(t,y) = \limsup_{\substack{(\Delta t,\Delta y,M_Y)\to(0,1,\infty)\\(t',y')\to(t,y)}} V^{\Delta t,\Delta y,M_Y}(t',y'),$$

$$V_*(t,y) = \liminf_{\substack{(\Delta t,\Delta y,M_Y)\to(0,1,\infty)\\(t',y')\to(t,y)}} V^{\Delta t,\Delta y,M_Y}(t',y'),$$

which are finite upper and lower semi-continuous functions on $[0,1]\times\mathbb{Z}$, and inherit the boundedness of $(V^{\Delta t,\Delta y,M_Y})$, are viscosity sub and super solutions of (4.2.13) respectively. If we assume that the claim is true, we can obtain $V^*\le V_*$ by the strong comparison principle for (4.2.13). Since the converse inequality is obvious by the definitions of V^* and V_*, we can show that $V^*=V_*=V$ is the unique bounded continuous viscosity solution to (4.2.13), hence completing the proof of convergence.

Next we prove the viscosity supersolution property of V_*. The viscosity subsolution property of V^* can be proved analogously. We introduce $(\bar t,\bar y)\in[0,1)\times\mathbb{Z}$ and let $\psi\in C^1_b$ be a test function such that $(\bar t,\bar y)$ is a strict global minimum point of $V_*-\psi$, i.e.

$$0 = (V_*-\psi)(\bar t,\bar y) = \min_{(t,y)\in[0,1)\times\mathbb{Z}} (V_*-\psi)(t,y). \tag{4.4.6}$$

By definition of $V_*(\bar t,\bar y)$, there exists a sequence (t'_n,y'_n) in $[0,1)\times\mathbb{R}$, and a sequence $(\Delta t_n,\Delta y_n,M_Y^n)$ such that

$$(t'_n,y'_n)\to(\bar t,\bar y),\quad (\Delta t_n,\Delta y_n,M_Y^n)\to(0,1,\infty)\quad\text{and}\quad V^{\Delta t_n,\Delta y_n,M_Y^n}(t'_n,y'_n)\to V_*(\bar t,\bar y),$$

when n goes to infinity. By the continuity of ψ and (4.4.6), we also have that

$$\zeta_n := (V^{\Delta t_n,\Delta y_n,M_Y^n}-\psi)(t'_n,y'_n)\to 0,$$

when n goes to infinity. By the definition of ζ_n, we have $V^{\Delta t_n,\Delta y_n,M_Y^n}\ge\psi+\zeta_n$. From the definition of the numerical scheme $\mathcal{S}^{\Delta t,\Delta y,M_Y}$, and its monotonicity, we then have

$$\begin{aligned}
\zeta_n+\psi(t'_n,y'_n) &= V^{\Delta t_n,\Delta y_n,M_Y^n}(t'_n,y'_n)\\
&= \mathcal{S}^{\Delta t_n,\Delta y_n,M_Y^n}(t'_n+\Delta t_n,y'_n,V^{\Delta t_n,\Delta y_n,M_Y^n},p^{\Delta t_n,\Delta y_n,M_Y^n})\\
&\ge \mathcal{S}^{\Delta t_n,\Delta y_n,M_Y^n}(t'_n+\Delta t_n,y'_n,\zeta_n+\psi,p^{\Delta t_n,\Delta y_n,M_Y^n})\\
&= \mathcal{S}^{\Delta t_n,\Delta y_n,M_Y^n}(t'_n+\Delta t_n,y'_n,\psi,p^{\Delta t_n,\Delta y_n,M_Y^n})+\zeta_n
\end{aligned}$$

$$= \psi(t'_n + \Delta t_n, y'_n) + \Delta t_n \times \hat{H}(t'_n + \Delta t_n, y'_n, \psi, p^{\Delta t_n, \Delta y_n, M_Y^n}) + \zeta_n,$$

which implies

$$\frac{\psi(t'_n, y'_n) - \psi(t'_n + \Delta t_n, y'_n)}{\Delta t_n} - \hat{H}(t'_n + \Delta t_n, y'_n, \psi, p^{\Delta t_n, \Delta y_n, M_Y^n}) \geq 0.$$

By the consistency property combined with Assumption 4.4.1, and by sending n to infinity in the above inequality, we obtain the required viscosity supersolution property

$$-\psi_t(\bar{t}, \bar{y}) - H(\bar{t}, \bar{y}, \psi, p) \geq 0. \qquad \square$$

4.4.3 Numerical optimal trading strategy

Now we list all of numerical results of the strategic trader's optimal strategies in different scenarios listed in Table 4.2.

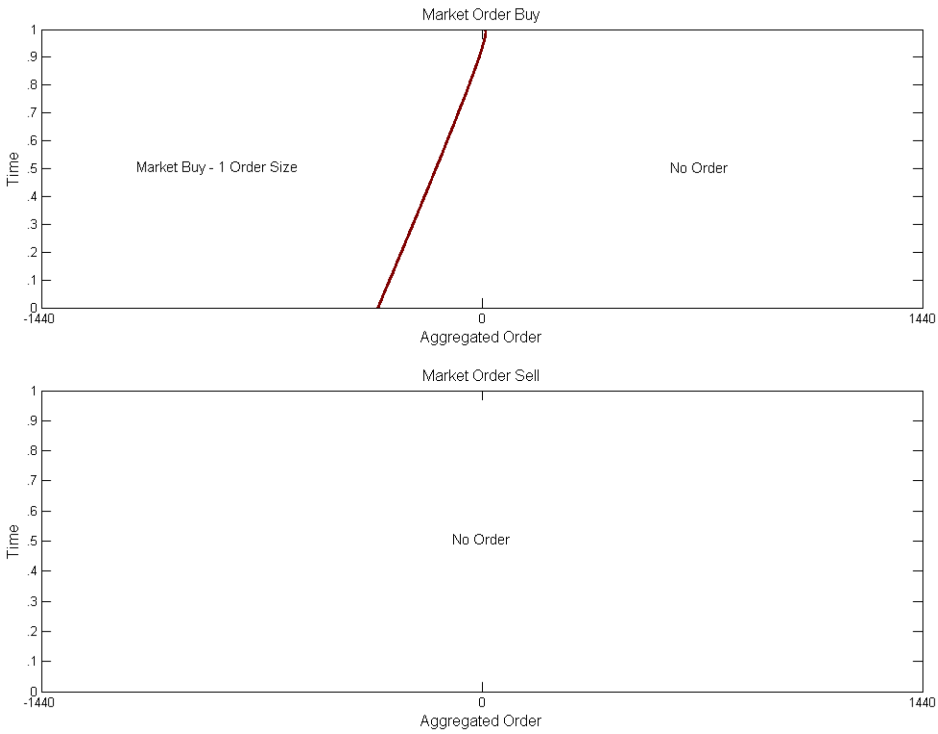


Figure 4.3: Optimal strategy of 1 market order size only

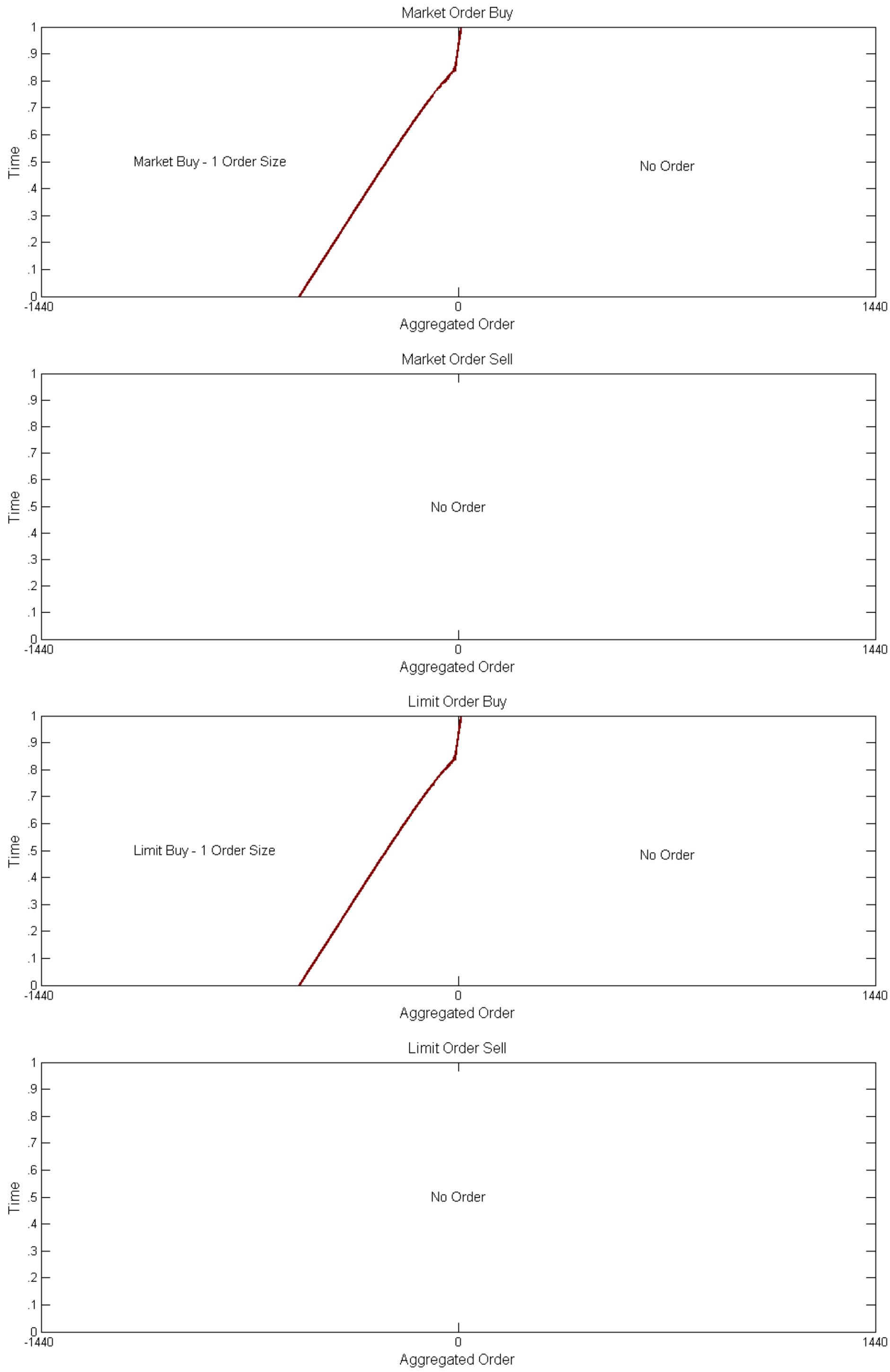


Figure 4.4: Optimal strategy of 1 market order size and 1 limit order size

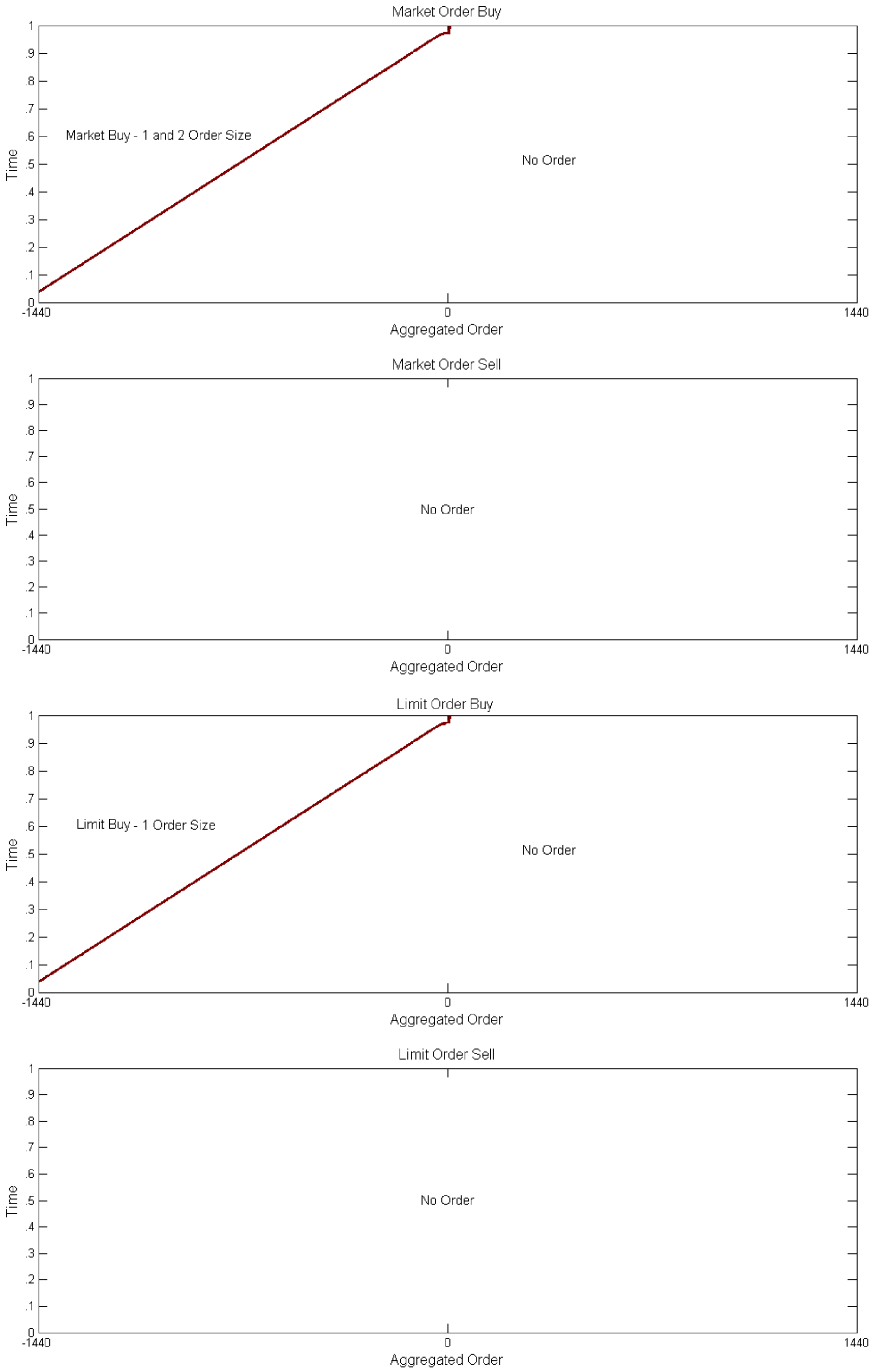


Figure 4.5: Optimal strategy of 2 market order size and 1 limit order size

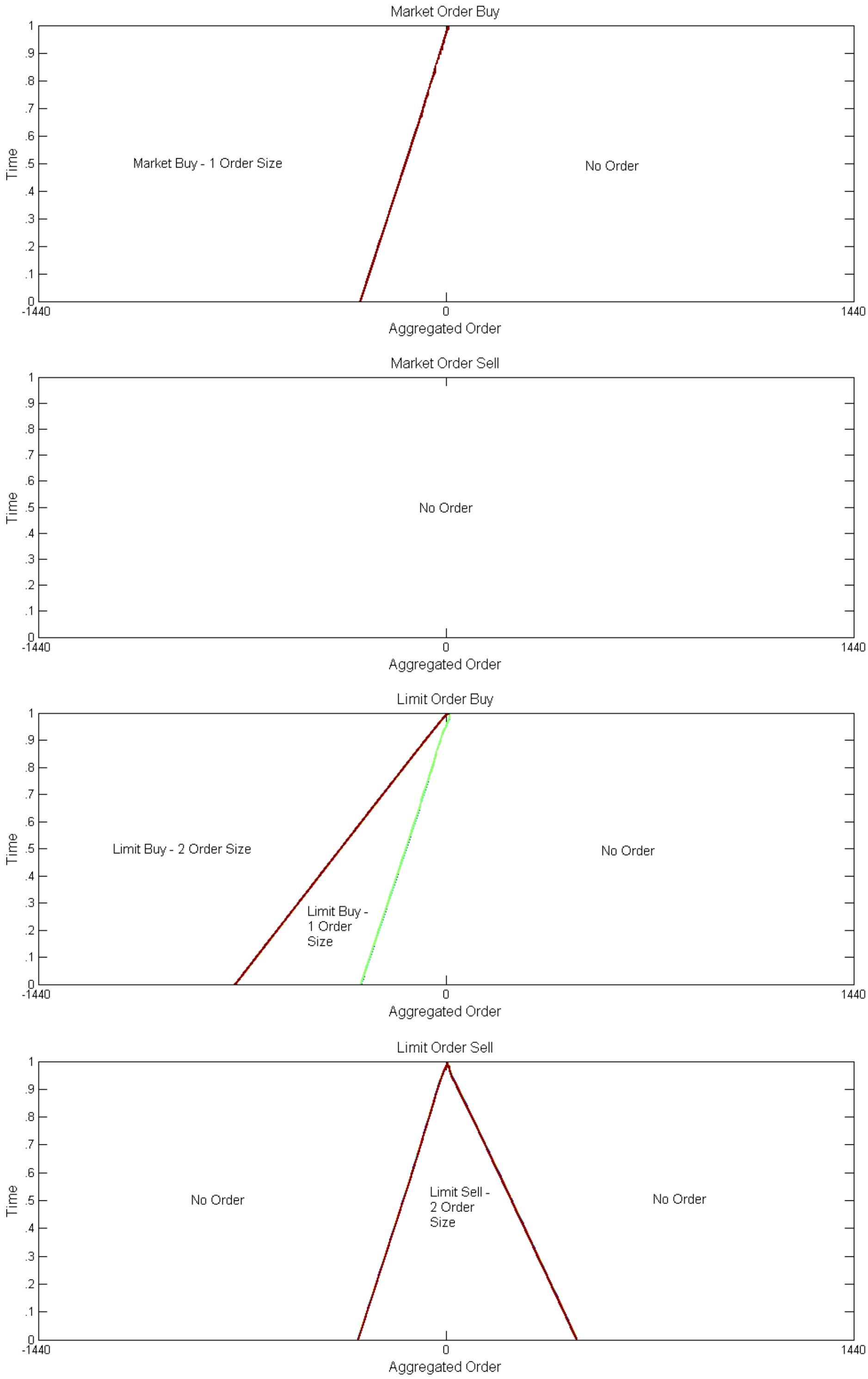


Figure 4.6: Optimal strategy 1 market order size only and 2 limit order size

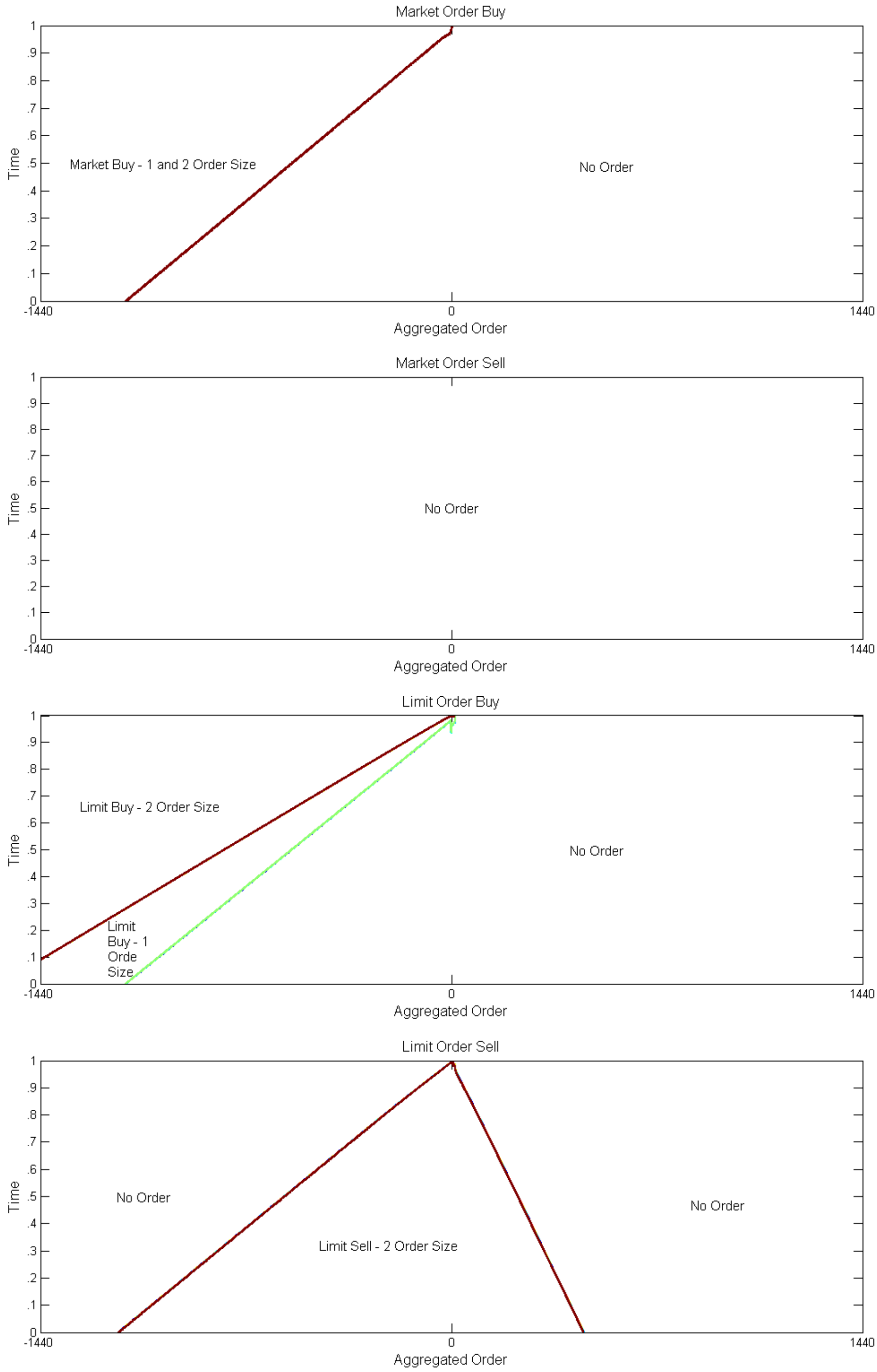


Figure 4.7: Optimal strategy 1 market order size only and 2 limit order size

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